Physics 602 **Homework Set 5** **Spring, 2008**

Due date: Monday April 7

Reading: Ziman Ch. 10 and optionally, Kaxiras Ch. 7

1) (25 points) Holstein-Primakoff (H-P) transformation.
We search for a bosonic operator $a_j^\dagger$ that essentially decrements the spin on site j, in terms of which we will eventually write the magnon creation operator as

$$b_{\mathbf{k}}^\dagger = \frac{1}{\sqrt{N}} \sum_j e^{-i\mathbf{k}\cdot\mathbf{R}_j}\, a_j^\dagger$$

We will insist that the a_j have the usual Bose commutation relations $[a_i\,, a_j^\dagger] = \delta_{ij}$, $[a_i\,, a_j] = 0$. Let's focus on just one site for a while, temporarily dropping the subscript j. Then the a's obey the algebra

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a\ |n\rangle = \sqrt{n}\,|n-1\rangle$$

while the spin operators (for spin quantum number 's') obey the algebra

$$S_Z|m\rangle = m|m\rangle$$

$$S_+|m\rangle = \sqrt{s+m+1}\,\sqrt{s-m}\,|m+1\rangle$$

$$S_-|m\rangle = \sqrt{s+m}\,\sqrt{s-m+1}\,|m-1\rangle$$

a) We want to identify the maximally spin-up state $|m = s\rangle$ with the boson "vacuum" $|n = 0\rangle$, and more generally, to identify $|m\rangle = |n\rangle$ where $m = s - n$. Show that this can be accomplished by the (rather unlikely-looking) H-P transformation

$$S_z = s - a^\dagger a$$

$$S_+ = [2s - a^\dagger a]^{1/2}\, a$$

$$S_- = a^\dagger\, [2s - a^\dagger a]^{1/2}$$

b) Is this transformation faithful? We might worry that the spin representation $|m\rangle$ is $(2s+1)$-dimensional, while the Boson representation is infinite-dimensional. Show, however, that any Hamiltonian written using the substitions S_z, $S_+\,, S_- \rightarrow a$, $a^\dagger$ does not mix the block of states $|n\rangle = |0\rangle$, ..., $|2s\rangle$ with the higher states $|n\rangle = |2s+1\rangle$, $|2s+2\rangle$, ...

c) Restoring the crystal labels a_j and $b_{\mathbf{k}}$, show that the H-P transformation of the Heisenberg Hamiltonian leads to a Taylor expansion in s/N whose first two terms are

$$H = -JNZs^2 + 2JZs \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}})\, b^\dagger_{\mathbf{k}}\, b_{\mathbf{k}}$$

where $\gamma_{\mathbf{k}} = Z^{-1} \sum_\delta e^{i\mathbf{k}\cdot\delta}$ and Z is the number of nearest neighbors at distance δ. Also show, at least by a rough argument, that the p'th term in this expansion has a prefactor scaling like $s^{(p-3)}/N^{(p-2)}$ and with a pattern of operators $b^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2} b_{\mathbf{k}_3} b_{\mathbf{k}_4}$ (p=3), $b^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2} b^\dagger_{\mathbf{k}_3} b_{\mathbf{k}_4} b_{\mathbf{k}_5} b_{\mathbf{k}_6}$ (p=4), etc. (These terms with $p > 2$ describe the "magnon-magnon interactions" similarly to the way that anharmonic terms in the phonon Hamiltonian describe phonon-phonon interactions.)

d) Show (without going into detailed algebra) that the H-J transformation of the Ising Hamiltonian gives a spin-wave Hamiltonian of a similar form except that terms of order higher than $p = 3$ are rigorously absent.

Note: While we do not pursue the matter here, the H-P transformation is also useful for treating spin waves in antiferromagnets.

2) (15 points)
Consider the magnon-phonon Hamiltonian

$$H = \sum_k \left\{ \omega^m_k \, a^\dagger_k a_k \; + \; \omega^p_k \, b^\dagger_k b_k \; + \; c_k \, (\, a^\dagger_k b_k + a_k b^\dagger_k \,) \right\} \quad ,$$

where c_k is the coupling coefficient and $(a^\dagger, a)$ and $(b^\dagger, b)$ are magnon and phonon creation and annihilation operators. Show that the transformations

$$a_k = A_k \cos\theta_k + B_k \sin\theta_k \quad ,$$

$$b_k = B_k \cos\theta_k - A_k \sin\theta_k \quad ,$$

with θ real, diagonalize the Hamiltonian if

$$\tan 2\theta_k = \frac{2c_k}{\omega^p_k - \omega^m_k} \quad .$$

Sketch what the dispersion relations for the mixed excitations created by $A^\dagger_k$ and $B^\dagger_k$ ("phmagnons"? "maphnons"?) might look like in a ferromagnetic Bravais crystal, assuming that $c_k \propto k^2$ for small k (and that the bands cross before reaching the Brillouin zone edge).