Physics 602 **Homework Set 4** **Spring, 2008**

Due date: Monday March 24.

1) (25 points)
We generalize some of the discussion of $S(q,\omega)$ to finite temperature. Let n label the exact many-body eigenstates; at $\beta = 1/k_B T$, the thermal expectation value of an operator A is

$$\langle A \rangle = Z^{-1} \sum_n e^{-\beta\omega_n} \langle n|A|n\rangle \ \ ; \ \ Z = \sum_n e^{-\beta\omega_n} \ .$$

We define

$$S(q,t) = \langle \rho_q(t) \rho_q^\dagger(0) \rangle$$

and the expression for $\epsilon^{-1}(q,\omega)$ is also written as a weighted sum over initial states. Show:

a) $S(q,\omega) = e^{\beta\omega} S(q,-\omega)$;
b) $\mathrm{Im}\, \epsilon^{-1}(q,\omega) = -\pi \tilde{V}(q) \left[1 - e^{-\beta\omega}\right] S(q,\omega)$.

2) (30 points)
The purpose of this problem is to work out an *exact* epression for the *transverse* dielectric function of the uniform electron gas, using the Kubo-formula approach. The transverse dielectric function is defined as

$$\epsilon_{\alpha\beta}(q,\omega) = \delta_{\alpha\beta} + \frac{4\pi i \sigma_{\alpha\beta}(q,\omega)}{\omega}$$

where the conductivity σ is defined via

$$J_\alpha(r,t) = \int dr' \int dt' \ \sigma_{\alpha\beta}(r-r', t-t')\, E_\beta(r',t') \ ,$$

$$J_\alpha(q,\omega) = \sigma_{\alpha\beta}(q,\omega)\, E_\beta(q,\omega)$$

(implied sum notation used throughout). In the Coulomb gauge the perturbation is

$$H'(t) = -\frac{1}{c} \int dr\, j_\alpha(r)\, A_\alpha(r,t) \ .$$

Now there is a subtlety: J_α and j_α are not quite the same; they are defined as

$$j_\alpha(r) = \frac{e}{2m} \sum_i \left[p_{i\alpha}\, \delta(r-r_i) \ + \ \delta(r-r_i)\, p_{i\alpha} \right] \ ,$$

$$J_\alpha(r) = \frac{e}{2} \sum_i \left[v_{i\alpha}\, \delta(r-r_i) \ + \ \delta(r-r_i)\, v_{i\alpha} \right] \ .$$

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Recalling that $p = mv + \frac{e}{c}A$, it follows that

$$\langle\, j_\alpha(r) \;-\; J_\alpha(r)\,\rangle \;=\; \frac{e^2 n}{mc}\, A_\alpha(r)$$

where $\langle\rangle$ is evaluated in the ground state of uniform density. Finally, recall that $E = (-1/c)(dA/dt)$ so that $E(q,\omega) = i\omega A(q,\omega)/c$.

a) Using the Kubo approach derived in class, with a source acting on operator $j_\alpha(r', t')$ and a response obtained from the expectation of $j_\beta(r,t)$, show that

$$\sigma_{\alpha\beta}(q,\omega) = \frac{i}{\omega}\,[\,\Pi_{\alpha\beta}(q,w) \;+\; \frac{ne^2}{m}\delta_{\alpha\beta}\,]$$

where

$$\Pi_{\alpha\beta}(q,t) = -i\,\theta(t)\,\langle\, j_\alpha(q,t)\,,\, j_\beta^\dagger(q,0)\,\rangle \;\; .$$

b) Show that this leads to a transverse dielectric function

$$\epsilon_{\alpha\beta}(q,\omega) \;=\; \left[\,1 - \frac{\omega_p^2}{\omega^2}\,\right]\delta_{\alpha\beta} \;-\; \frac{4\pi}{\omega^2}\,\Pi_{\alpha\beta}(q,\omega) \;\; .$$

Note that the result is non-trivial even in the extreme approximation $\Pi = 0$.

c) Write down the exact spectral representation of $\Pi_{\alpha\beta}(q,\omega)$ in terms of the exact many-body eigenstates $|n\rangle$ and their excitation energies ω_{n0}.

3) (25 points)
This is a simple exercise in Fermi liquid theory. Starting from

$$\delta E = \sum_k \epsilon(k)\,\delta f(k) \;+\; \frac{1}{2V}\sum_k\sum_{k'} u(k,k')\delta f(k)\delta f(k') \;\; ,$$

$$E(k) = \frac{\delta E}{\delta f(k)} \;\; ,$$

$$\mathbf{v}^{(0)}(k) = \nabla_k \epsilon(k) \;\; ,$$

$$\mathbf{v}(k) = \nabla_k E(k) \;\; ,$$

show that the contribution to the particle current

$$J = \sum_k f(k)\mathbf{v}(k)$$

from the quasiparticle k is, at $T = 0$,

$$\mathbf{j}_k = \mathbf{v}^{(0)}(k) + \frac{1}{V}\sum_{k'} u(k,k')\,\mathbf{v}^{(0)}(k')\,\delta[\epsilon(k') - \epsilon_F]$$

where ϵ_F is the Fermi level. Hint: An integration by parts may be needed at some stage.