Physics 602 **Homework Set 3** **Spring, 2008**

Due date: Monday March 3.

1) (10 points)
Consider the expression

$$\widetilde{E}(\mathbf{k}) \;=\; E(\mathbf{k}) \;+\; \sum_{\mathbf{q}} |M_{\mathbf{q}}|^2 \,\frac{1 - 2\bar{n}_{\mathbf{k+q}}}{E(\mathbf{k}) - E(\mathbf{k}+\mathbf{q}) - \omega_{\mathbf{q}}} \quad ,$$

derived in class for the renormalization of the electron quasiparticle energy when the electrons (k) are coupled to plasmons (q). Argue that the renormalization leads to a *reduction* in the Fermi velocity $dE/dk|_{k_F}$. You may assume that the dominant contributions come from small q. [This is intuitive in a way: the electron gets heavier and slower when dressed by a cloud of plasmons.]

2) (30 points)
In class, I introduced (without derivation – that is coming shortly) the expression for the Lindhard dielectric susceptibility

$$\chi(\mathbf{q}, \omega) = -\frac{e^2}{\Omega} \sum_{\mathbf{k}\sigma} \frac{\bar{n}_{\mathbf{k}} \; - \bar{n}_{\mathbf{k+q}}}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \omega + i\eta}$$

where $\bar{n}_{\mathbf{k}} \; = \Theta(k_F - k)$ and η is an infinitessimal which is taken to zero from above. In the present problem, we only consider real parts of the response functions, and therefore discard η.

a) Show that (the real part of) χ can be written

$$\chi(q, \omega) = -\frac{2e^2}{\Omega} \sum_{\mathbf{k}\sigma} \bar{n}_{\mathbf{k}} \;\; \frac{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k})}{[E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k})]^2 - \omega^2} \quad .$$

[Hint: Write the terms involving $\bar{n}_{\mathbf{k}}$ and $\bar{n}_{\mathbf{k+q}}$ separately, and change the sum index from $\mathbf{k}$ to $\mathbf{k}' = \mathbf{k} + \mathbf{q}$ in the latter.]

b) Show that in the long-wavelength limit, $q \to 0$, one obtains

$$\epsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \quad ,$$

where ω_{p} is the plasma frequency.

c) Show that in the static limit, $\omega \to 0$, one obtains $\chi(q) = -e^2 n(\epsilon_F)\, F(x)$, with

$$F(x) \;=\; \left[\frac{1}{2} + \frac{1 - x^2}{4x} \, \ln \left| \frac{1+x}{1-x} \right| \right] \quad ,$$

and $x = q/2k_F$.

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3) (20 points)

Kramers-Kronig ("KK") relations:

a) Show that if the kernel $K(t-t')$ appearing in

$$B(t) = \int dt'\, K(t-t')\, A(t')$$

is not only causal but real, then the KK relations take the form

$$K_1(\omega) = \frac{2}{\pi} \int_0^\infty K_2(\omega')\; P\, \frac{\omega' d\omega'}{(\omega')^2 - \omega^2} \quad ,$$

$$K_2(\omega) = -\frac{2}{\pi} \int_0^\infty K_1(\omega')\; P\, \frac{\omega\, d\omega'}{(\omega')^2 - \omega^2} \quad ,$$

where K_1 and K_2 are the real and imaginary parts of K respectively.

b) For a system with translational and rotational invariance, the response function $K(q,t)$ (i.e., with *wavevector* and *time* arguments) is indeed real. (Why?) So, using (a) together with $\epsilon(q,\omega) = 1 - \tilde{V}(q)\chi(q,\omega)$ and $\epsilon^{-1}(q,\omega) = 1 + \tilde{V}(q)\pi(q,\omega)$, derive KK relations for $\epsilon_1(q,\omega)$, $\epsilon_2(q,\omega)$, $\mathrm{Re}\,\epsilon^{-1}(q,\omega)$, and $\mathrm{Im}\,\epsilon^{-1}(q,\omega)$, involving integrals over only half the frequency axis.