Bogoliubov-Valatin Formalism

Forward and reverse transform between γ's and c's:

$$\boxed{\begin{array}{ll} \gamma_k^\dagger = u_k\, c_k^\dagger - v_k\, c_{-k} & \qquad c_k^\dagger = u_k\, \gamma_k^\dagger + v_k\, \gamma_{-k} \\ \gamma_{-k}^\dagger = u_k\, c_{-k}^\dagger + v_k\, c_k & \qquad c_{-k}^\dagger = u_k\, \gamma_{-k}^\dagger - v_k\, \gamma_k \end{array}}$$

It is straightforward to check that the γ's obey the usual Fermion anticommutation relations. Define also

$$\eta_k = \gamma_k^\dagger \gamma_k$$

Transform from c's to γ's:

$$\begin{aligned} H - \mu N = \textstyle\sum_k \tilde{\epsilon}_k \, [\, 2v_k^2 + (u_k^2 - v_k^2)(\eta_k + \eta_{-k}) + 2u_k v_k (\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k}\gamma_k) \,] \qquad & \text{(i)} \\ -V \textstyle\sum_{kk'} [\, u_k v_k u_{k'} v_{k'} (1 - \eta_{k'} - \eta_{-k'})(1 - \eta_k - \eta_{-k}) \qquad & \text{(ii)} \\ +(u_k^2 - v_k^2) u_{k'} v_{k'} (1 - \eta_{k'} - \eta_{-k'})(\gamma_{-k}\gamma_k + \gamma_k^\dagger \gamma_{-k}^\dagger) \qquad & \text{(iii)} \\ +(u_k^2 \gamma_{-k}\gamma_k - v_k^2 \gamma_k^\dagger \gamma_{-k}^\dagger)(u_{k'}^2 \gamma_{k'}^\dagger \gamma_{-k'}^\dagger - v_{k'}^2 \gamma_{-k'}\gamma_{k'}) \,] \qquad & \text{(iv)} \end{aligned}$$

Approximate $\eta_k \to \bar{\eta}_k = \langle \psi | \eta_k | \psi \rangle$ (thermal and quantum expectation; depends on T).

Drop terms (iv).

Require that $\gamma^\dagger \gamma^\dagger$ and $\gamma\gamma$ terms should vanish [line (iii) and second part of (i)]:

$$0 = 2\tilde{\epsilon}_k u_k v_k - (u_k^2 - v_k^2) V \textstyle\sum_{k'} u_{k'} v_{k'} (1 - \bar{\eta}_{k'} - \bar{\eta}_{-k'})$$

Define $\Delta = V \sum_k u_k v_k (1 - \bar{\eta}_k - \bar{\eta}_{-k})$ or, using $\bar{\eta}_k = \bar{\eta}_{-k} = [\exp(E_k/k_B T) + 1]^{-1}$,

$$\boxed{\Delta = V \sum_k u_k v_k \tanh\left(\frac{E_k}{2k_B T} \right)}$$

where $E_k = \sqrt{\tilde{\epsilon}_k^2 + \Delta^2}$. Then $u_k = \cos(\theta_k/2)$ and $v_k = \sin(\theta_k/2)$ are given by

$$\boxed{\tan\theta_k = \frac{\Delta}{-\tilde{\epsilon}_k}}$$

Substituting, the finite-T BCS gap equation becomes

$$\boxed{1 = \frac{1}{2} V g_f \int_0^{\omega_D} d\epsilon \, \frac{\tanh(\sqrt{\epsilon^2 + \Delta^2}/2k_B T)}{\sqrt{\epsilon^2 + \Delta^2}}}$$

and its solution gives $\Delta(T)$.