An introduction to generalized vector spaces and Fourier analysis.
by M. Croft

FOURIER ANALYSIS : Introduction Reading: Brophy p. 58-63
This lab is u lab on Fourier analysis and consists of VI parts. In part I the conventional three dimensional vector space is reviewed In part II the general concept of a function as a vector
in a vector space of functions is developed in strict analogy to the review in part I. In section III Fourier analysis is introduced as a specific example of the general function vector space concepts of section II.
These first three sections are optional material intended as a painless first introduction of the professional physics major to .mathematical concepts which will appear again and again in future course work. You are encouraged to read these first three sections but do-not-become bogged down in them,
Section IV begins the practical part of this lab. The definitions necessary to expand a function in a Fourier series are given along with a helpful note on odd and even *functions.* Section V consists of several worked-out examples with Questions to be answered. Section VI consists of experimental verification of some of the: examples worked out in section V.

I. Three dimensional space review

Consider the three dimensional vector space familiar to us from introductory physics. Any vector A in this space can be written as

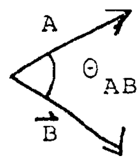
$$\vec{A} = A_1\hat{X}_1 + A_2\hat{X}_2 + A_3\hat{X}_3 \qquad (1)$$

Where Ai, is the i'th component of A along the X_i'th axis and x_i is the unit vector along the x_i direction. (Note A_1 is just a number). The fact that any vector in this vector space can be written as the sum (1) means that the three vectors Xi; i=1,3 form a complete set of unit vectors for this vector space.

Using the usual dot product (or inner product, or scalar product) in this vector space i.e.

$$\vec{A} \cdot \vec{B} = |A|\,|B| \cos \Theta_{AB} \qquad (2)$$

where



one finds the unit vectors obey the relation

$$\hat{X}_i \cdot \hat{X}_j = \delta_{ij}$$

or

$$(\hat{X}_i, \hat{X}_j) = \delta_{ij} \qquad (3)$$

Here the second notation $(\hat{x}_i, \hat{x}_j) = \hat{x}_i \bullet \hat{x}_j$ is a more general notation for an inner product and $\delta_{i.j}$ is the "Kronicker delta" symbol defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (4)$$

Relation 3 is a statement that the unit vectors are orthogonal (i.e. θ_{AB}= 90° in the sense of (2)) and are normalized to unit length i.e. $|\hat{x}_i|^2 = 1$. (These two terms are sometimes combined and the unit vectors are called orthonormal).

One can project out the component of A along the x_i'th axis via the inner product

$$\vec{A}\cdot\hat{x}_i = \left(\sum_{j=1}^{3} A_j \hat{x}_j\right)\cdot\hat{x}_i$$
$$= \sum_j A_j \hat{x}_j\cdot\hat{x}_i = \sum_j A_j \delta_{ij} \tag{5}$$
$$= A_i \quad .$$

This relation can be used as a definition of the i'th component of $\vec{A}$ (i.e. $A_i = \hat{x}_i \bullet \vec{A}$)
The length of any vector is defined in terms of the dot product via

$$\sqrt{|\vec{A}|^2} = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{6}$$

The dot product of two vectors can also be expressed in terms of the components of the two vectors via

$$\vec{A}\cdot\vec{B} = \left(\sum_i A_i \hat{x}_i\right)\cdot\left(\sum_j B_j \hat{x}_j\right) = \sum_{ij} A_i B_j (\hat{x}_i\cdot\hat{x}_j)$$
$$= \sum_{ij} A_i B_j \delta_{ij} = \sum_{i=1}^{3} A_i B_i \tag{7}$$

The key points of this review which we will generalize are
(1) The existence of a vector space
(2) The existence of *a* complete set of unit vectors in terms of which any vector in the vector space can be represented
(3) The exi5tance of *an* inner product (dot product, or scalar product) with respect to which the unit vectors are orthogonal and of unit length.

II. Vector space of functions (general)
The set of all possible functions f(x) on an interval of x between L_1 to L_2 forms a vector space. (Some extra conditions on these functions will be specified later *in t'..e* Fourier series example). It so happens that the complete set of unit vectors for such a vector space are an infinite set of functions

$$\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_i(x), \ldots \}$$

The *expansion of a* function f(x) in terms *of* these unit vectors *is now* (analogy to (1)

$$f(x) = \sum_{i=1}^{\infty} f_i \, \varphi_i(x) \qquad \textit{(II-1\}}$$

*W*here f_i is just a number and represents the *component* of f (x) along the "direction" defined by the unit vector $\varphi_i(x)$.

The inner product of two vectors in this vector space is defined as the integral of the product of the two functions. Over the interval $[L_1,L_2]$, the analogy to 3 in this vector space. is therefore

$$(\varphi_i, \varphi_j) = \int_{L_1}^{L_2} \varphi_i(x)\, \varphi_j(x)\, dx = \delta_{ij} \qquad \text{(II-3)}$$

which tells us that the unit vectors are <u>orthonormal,</u> (i.e.. orthogonal to each other and of unit length).

One can project out the component of f(x) along the $\varphi_i(x)$ axis via this inner product as follows

$$(f, \varphi_j) = \int_{L_1}^{L_2} f(x)\, \varphi_j(x)\, dx = \sum_i f_i \int_{L_1}^{L_2} \varphi_i(x)\, \varphi_j(x)\, dx \qquad \text{(II-5)}$$

$$= \sum_i f_i\, \delta_{ij} = f_j \qquad \text{(II-5}$$

As before the I'th component of f(x) can be expressed in terms of the above as

$$f_i = \int_{L_1}^{L_2} f(x)\, \varphi_i(x)\, dx$$

The length squared of any vector f(x) in this vector space in terms of the inner product is

$$(f, f) = \int_{L_1}^{L_2} f^2(x)\, dx = \sum_{ij} f_i f_j \int_{L_1}^{L_2} \varphi_i(x)\, \varphi_j(x)\, dx$$

$$= \sum_{i=1}^{\infty} f_i^2 \qquad (:$$

The inner product of two vectors f(x) and g(x) can beexpressed in terms of their components via

$$(f,g) = \int_{L_1}^{L_2} f(x)\, g(x)\, dx = \sum_{ij} f_i\, g_j \int_{L_1}^{L_2} \varphi_i(x)\, \varphi_j(x)\, d. \tag{II-7}$$

$$= \sum_{ij} f_i\, g_j\, \delta_{ij} = \sum_{i=1}^{\infty} f_i\, g_i$$

Finally we should emphasize in what mathematical sense the representation II-1 is true. Suppose we cut short the infinite sum in (II-1) after some finite number (N) of terms and then subtract this finite sum from f(x)

i. e.' $$\Delta_N f(x) = f(x) - \sum_{i=1}^{N} f_i\, \varphi_i(x)$$

The difference $\Delta_N f(x)$ is a vector in the vector space of functions. The relation (II-1) is said to be true in the sense that the "length" of the difference function $\Delta_N f(x)$ is small and in fact goes to zero as $N \rightarrow \infty$. That is

$$\lim_{N \to \infty} \int_{L_1}^{L_2} [\Delta_N f(x)]^2 dx \longrightarrow 0$$

III Fourier Series

Now we shall consider a specific example of the Fourier series. Our vector space is the space of <u>all functions on the interval</u>

$$-\pi \leq x \leq \pi.$$

(Actually for mathematical reasons we require that all the functions in our vector space must;

(1) be square -integrable, $\int_{-\pi}^{\pi} f^2(t)\, dt < \infty$

and (2) have at most a finite number of finite jump discontinuities.
Note: (Condition (1) means that we consider vectors of finite length only) The complete set of unit vector functions on this interval are

$$\left\{ \begin{array}{ll} \frac{1}{\sqrt{\pi}} \sin(mt) & m = 1, 2, \ldots \infty \\ 1/\sqrt{\pi} \cos(mt) & m = 1, 2, \ldots \infty \\ 1/\sqrt{2\pi} & \end{array} \right\}$$

(note: Both sin and cos terms are needed for this set to be complete).
The representation of f(t) in terms of these unit vectors is then called a Fourier series

$$f(t) = \sum_{m=1}^{\infty} \left[f_m^s \frac{\sin(mt)}{\sqrt{\pi}} + f_m^c \frac{\cos(mt)}{\sqrt{\pi}} \right] + f_0$$

The inner product of two of these unit vectors is in this case

$$\int_{-\pi}^{\pi} \frac{\sin(mt)\sin(nt)}{\pi} = \delta_{mn}$$

$$\int_{-\pi}^{\pi} \frac{\sin(mt)\cos(nt)}{\pi} = \delta_{mn}$$

The projection of f(t) along a particular axis is then

$$\left(f(t), \frac{\sin(mt)}{\sqrt{\pi}}\right) = \int_{-\pi}^{\pi} f(t) \frac{\sin(mt)}{\sqrt{\pi}} dt$$

$$= f_m^s$$

and a similar relation holds for the projection on the M-th cosine axis.
The length squared of vectors in this space is just

$$(f(t), f(t)) = \int_{-\pi}^{\pi} f^2(t)\, dt$$

$$= f_0 + \sum_{m=1}^{\infty} \left[(f_m^c)^2 + (f_m^s)^2 \right]$$

The inner product of two functions in this vector space is

$$(f(t), g(t)) = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

$$= f_0 g_0 + \sum_{m=1}^{\infty} \left[f_m^s g_m^s + f_m^c g_m^c \right] \qquad \text{(III-7)}$$

IV Practical Fourier Series

i. definitions

This section begins our practical treatment of Fourier analysis. Consider a function f(t) which is (1) periodic with a period T, (2) has at most a finite number of finite step discontinuities over any period T, and (3) is square integrable over the period T. Then this function can be expanded in a Fourier series

$$f(t) = f_0 + \sum_{n=1}^{\infty} \left\{ f_n^c \cos(n\omega t) + f_n^s \sin(n\omega t) \right\} \qquad \text{(IV-1)}$$

Here $\omega = \frac{2\pi}{T} = 2\pi\nu$ the constant expansion coefficients f_o, f_n^c and f_n^s are given by

$$f_n^c = \frac{2}{T} \int_{-T/2}^{T/2} f(t') \cos(n\omega t')\, dt' \qquad \text{IV-2}$$

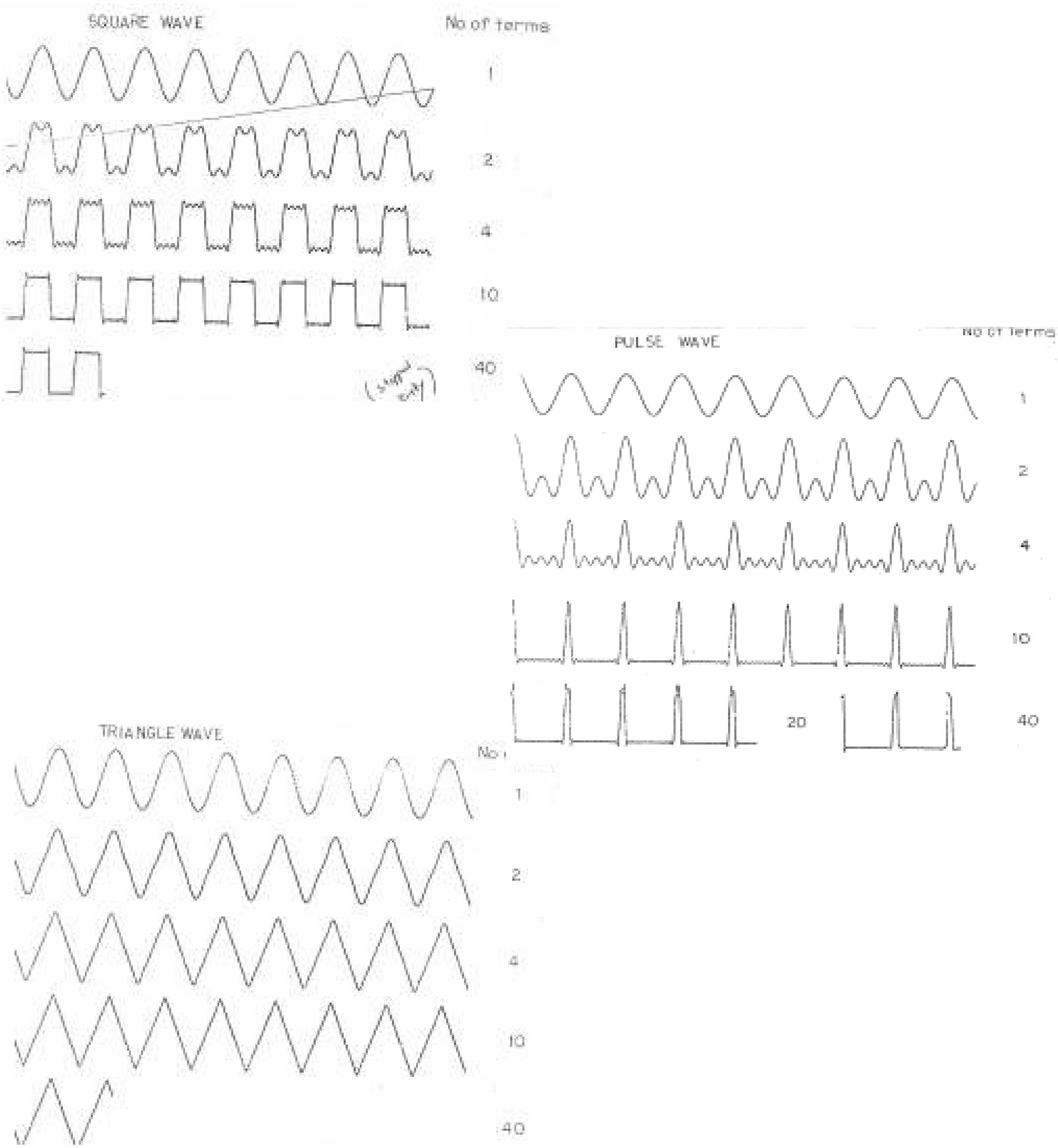
$$f_n^s = \frac{2}{T} \int_{-T/2}^{T/2} f(t') \sin(n\omega t')\, dt' \qquad \text{IV-3}$$

$$f_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t')\, dt' \qquad \text{IV-4}$$

ii) decomposition buildup

In words; if one has a function which is periodic with frequency $\nu = \frac{\omega}{2\pi}$, then this function can be written as an infinite sum of sine and cosine functions with frequencies equal to multiples of the original frequency ν.

One can alternatively think of being able to break down f(t) into a sum of sine waves or of being able to construct f(t) from a sum of sine saves. How a general function can be built up from sine waves is illustrated below where the sum of the first N= 1,?., 4, 10, 40 terms of several Fourier series have been plotted.



iii) <u>even and odd functions</u>

For a given function f(t) one can find the Fourier coefficients f_o, f_n^c and f_n^s by direct integration as in (Iv-2) (Iv-4). This task can be simplified with the following two theorems.

<u>Theorem.</u> If a function is an even function lie, f(-t) = f(t)] then its Fourier series contains only cosine terms.
<u>Theorem</u>. If a function is odd Lie, f(-t) = f{t)] then its Fourier series contains only sine terms.

These two Theorems are based simply or, the fact that <u>each</u> term of the Fourier series must have the same even-odd symmetry as the original function f(t).

Note: The definition of a function whether a function is odd with a (Fourier sine series), even (with a Fourier cosine series) or neither (with both cosine and sine terms in its Fourier series) is somewhat arbitrary. For example consider the square wave with the origin (t=o) as specified as below the function is odd and has a Fourier sine series.

$$f(t) = \begin{cases} h/2 & -\pi < t < 0 \ \ldots \\ -h/2 & 0 < t < \pi \ \ldots \end{cases}$$

$$f(t) = \frac{2h}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$$

Now translate the function by $\pi/2$ and one has

$$g(\tau) = \begin{cases} -h/2 & -\pi < \tau < -\pi/2 \\ h/2 & -\pi/2 < \tau < \pi/2 \\ -h/2 & \pi/2 < \tau < \pi \end{cases}$$

$$g(\tau) = f(t+\pi/2) = \frac{2h}{\pi} \sum_{n=1}^{\infty} \frac{\sin n(\tau+\pi/2)}{n}$$
$$= \frac{2h}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n}$$

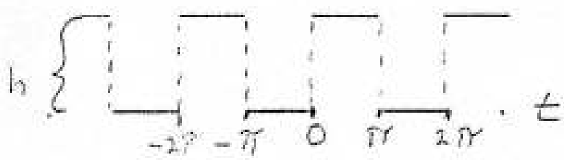
By translating this periodic function by $\pi/2$ it becomes an even function $[g(-\tau)=-g(\tau)]$ of the variable $\tau = t+\pi/2$ with a cosine Fourier series.

Thus a statement about the evenness or oddness of a function requires that the origin (t=0) be specified.

V Examples

Note: The following examples are worked out on the interval $-\pi < t < \pi$. To generalize to the interval $-T/2 < t < T/2$ one simply uses the transformation $\tau = t(T/2\pi) = t/\omega$

Square Wave.



$$f(t) = \begin{cases} 0 & -\pi < t < 0 \ldots \\ h & 0 < t < \pi \end{cases}$$

$$f_0 = \frac{1}{\pi} \int_0^{\pi} h \, dt = h$$

$$f_n^c = \frac{1}{\pi} \int_0^{\pi} h \cos(nt) \, dt = 0$$

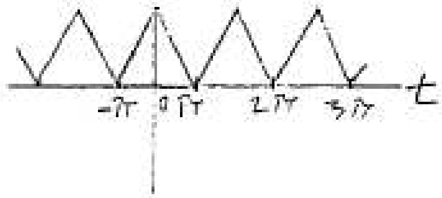
$$f_n^s = \frac{1}{\pi} \int_0^{\pi} h \sin(nt) \, dt = \frac{h}{n\pi} [1 - \cos n\pi]$$

$$f(t) = \underline{h} + \frac{2h}{\pi} \left[\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots \right]$$

Note:: The slow 1/n fall off in the higher harmonic terms. This is due to the step discontinuity which generates lots of high frequency terms.

Note: *The* series is a Sine series because the function is an odd function plus a constant. i.e.. take g(t) = f(t) - h/2 then g(-t) = - g(t) which is the definition of an odd function.

Triangle Wave



$$f(t) = \begin{cases} t & 0 < t < \pi \\ -t & -\pi < t < 0 \end{cases}$$

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1,3,5 \\ (n \text{ odd})}}^{\infty} \frac{\cos(nt)}{n^2}$$

a5)

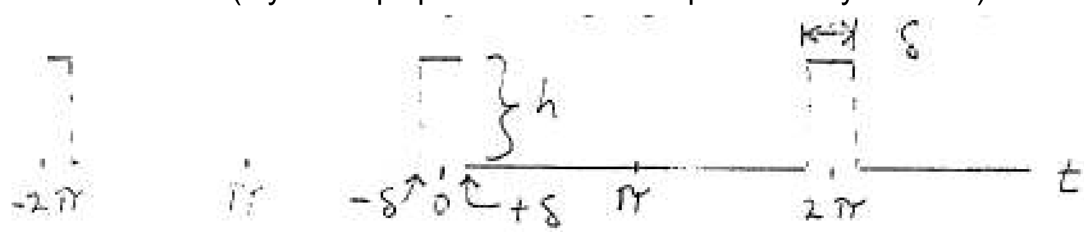
Note: This function is even i.e. f(t) = f(-t) and the series is a cosine series

Note: The $1/n^2$ fall off converges much more rapidly than the square wave due to the absence of jump disconuities.

Q1. How can you obtain the Fourier series of the triangle wave from that of the square wave?

Q2. How can you obtain the Fourier series of the square wave from that of the triangle wave?

Periodic Pulses (try in Exp. pulses of l0us separated by 100us.).



$$f(t) = \begin{cases} h & -\delta < t < +\delta \\ 0 & \text{for all other } t \end{cases}$$

This is an even function. Therefore it will have a cosine series,

$$f(t) = f_0 + \sum_{n=1}^{\infty} f_n^c \cos(nt)$$

$$f_n^c = \frac{1}{\pi} \int_{-\delta}^{\delta} h \cos(nt)\,dt$$

$$= \frac{1}{\pi} \left. \frac{h \sin nt}{n} \right|_{-\delta}^{\delta}$$

$$\Rightarrow$$

$$\boxed{f_n^c = \frac{2h}{\pi n} \sin(n\delta)}$$

Half-wave rectifier

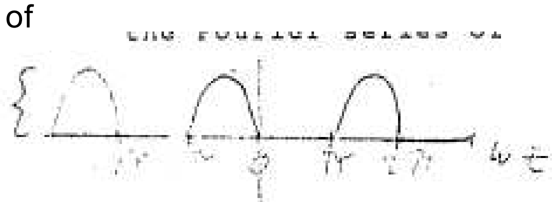
$$f(t) = \begin{cases} 0 & -\pi < \omega t < 0 \\ \sin \omega t & 0 < \omega t < \pi \end{cases}$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin(\omega t) - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty} \frac{\cos(n\omega t)}{n^2-1}$$

Q1. why are both sine and cosine terms present?

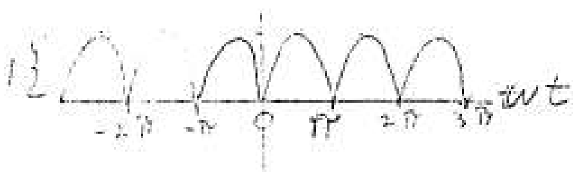
Q2. Why is the fall off not 1/n?

Q3. Show that by simple translation of the origin one can obtain the Fourier series of



$$g(t) = \begin{cases} \sin \omega t & -\pi < \omega t < 0 \\ 0 & 0 < \omega t < \pi \end{cases}$$

Full Wave Rectifier



$$f(t) = \begin{cases} -\sin(\omega t) & -\pi < \omega t < 0 \\ \sin(\omega t) & 0 < \omega t < \pi \end{cases}$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,\dots} \frac{\cos(n\omega t)}{n^2-1}$$

Q1. How can you obtain the result for the full wave rectifier from that of the half wave rectifier (Hint: One uses the principle of superposition which is true for all linear systems. In this case this means that the Fourier series for the sum of two functions is the sum of the separate Fourier series of the separate functions) .

VI Experimental

i) Procedure.

You will use a very accurate wave analyzer in this experiment. For your purposes this instrument can be thought of as an AC voltmeter with a tuned circuit on the input. This tuned circuit acts as a band pass filter allowing signals over a narrow frequency range to pass through to the voltmeter. You will be adjusting the center frequency of the band pass filter on the voltmeter input.

As you pass through a harmonic (multiple) of the fundamental frequency the meter will rise rapidly to a maximum value (at precisely the harmonic frequency) and then fall rapidly back to zero. You should record the amplitude and frequency of each component of the wave form. Record as many harmonics as you can find.

ii. Analysis for the Square Wave

1) Plot f_n versus the frequency $\nu = n\nu_o$ where ν_o is the fundamental frequency (note f_n is defined in formulas III-9 and III-11)

2.) Plot $\ln(\nu)$ versus n

3) From the plot in 2) above find, by a least squares fit or a BEBF (best-eye-ball-fit) to what extent the formula $\ln(f_n) = B \ln(\nu) + A$.

Compare your value of B with the theoretical value of B.

4) Use the value A and B determined in part 3 to plot a fit line through the data plotted in part 1).

iii. <u>Analysis for the triangle wave</u>

The analysis for the triangle wave is the same as that for the square wave. The theoretical value for B in the formula $f_n = B \ln \nu + A$ will of course be different for the triangle wave.

iv. <u>Analysis for the Pulse Wave</u>

1) Plot the Fourier amplitudes fn versus the frequency. Verify the periodic modulation of the amplitude and compare the observed period of the modulation to that expected from the theory and the pulse duration you used.
2) Fit a theoretical formula $f_\nu = (A/\nu) \sin(\delta\nu 2\pi)$ to your data.
To do this first recall that δ was determined in step 2). Next choose points at the peaks of the modulation (i.e. where $|\sin(\delta\nu 2\pi)| = 1$). Now plot these peak values of f_ν versus $1/\nu$ and extract A either by a least squares or best-eye-ball-fit. Sketch the theoretical curve on your plot of the data.

IV. Complex Fourier Series

The Fourier sine and cosine series you have been introduced to have the advantages of being real functions (of the sort you are most used to), and of making intuitively obvious the use of the odd sine and even cosine series for odd and even functions respectively. There is however, another mathematically more compact way to write a Fourier series based on De Moviers Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta \,.$$

To derive this alternate representation consider the following.

$$f(t) = \sum_{m=1}^{\infty}\left[f_m^s \frac{\sin(mt)}{\sqrt{\pi}} + \frac{f_m^c \cos(mt)}{\sqrt{\pi}}\right] + f_0$$

but.

$$\sin(mt) = \frac{e^{im} - e^{-imt}}{2i} \;;\quad \cos(mt) = \frac{e^{imt} + e^{-imt}}{2}$$

$$\Rightarrow \quad f(t) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{int}}{\sqrt{2\pi}}$$

where

$$f_n = \begin{cases} \dfrac{f_n^c - i f_n^s}{\sqrt{2}} & n > 0 \\[2ex] \dfrac{f_m^c + i f_m^s}{\sqrt{2}} & m = |n|,\ n < 0 \\[2ex] f_0 & n = 0 \end{cases}$$

or

$$f_n = \int_{-\pi}^{\pi} \frac{e^{-imt}}{\sqrt{2\pi}} f(t)\, dt$$

All of the general discussions presented earlier now hold for the complete set of functions

$$\left\{ \frac{e^{int}}{\sqrt{2\pi}} \; ; \; n = -\infty, \ldots, -1, 0, 1, \ldots, \infty \right\}$$

There is one important distinction however. The definition of the inner product must be changed to take account of the complex nature of the functions, i.e..

$$(\varphi_i , \varphi_j) = \int_{L_1}^{L_2} \varphi_i^+(x)\, \varphi_j(x)\, dx$$

where the φ_i^+ the complex conjugate of φ_i (that is i is replaced everywhere by –I The student should note that it is the above definition of the inner product which one uses in quantum mechanics.