Not required. For advanced students

Air friction effect

$$F = ma = -\frac{CA\rho}{2}v^2$$

$$\Downarrow$$

$$a = \frac{dv}{dt} = -\frac{1}{L}v^2 \qquad L = \frac{2m}{CA\rho}$$

$$\Downarrow$$

$$\frac{dv}{v^2} = -\frac{dt}{L}$$

$$\Downarrow$$

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\int_0^t \frac{dt}{L}$$

$$\Downarrow$$

$$\frac{1}{v} - \frac{1}{v_0} = -\frac{t}{L}$$

$$\Downarrow$$

$$\frac{v}{v_0} = \frac{1}{1 + \frac{v_0}{L}t}$$

$$\frac{1}{v_0}\frac{dx}{dt} = \frac{1}{1 + \frac{v_0}{L}t}$$

$$\int_0^x dx = L\int_0^t \frac{dt}{(\frac{L}{v_0} + t)}$$

$$\frac{x}{L} = \ln\left[\frac{(\frac{L}{v_0} + t)}{(\frac{L}{v_0})}\right]$$

$$\frac{x}{L} = \ln[(1 + t\frac{v_0}{L})]$$

$$e^{x/L} = (1 + t\frac{v_0}{L})$$

$$\frac{v}{v_0} = e^{-x/L}$$

Not required. For advanced students
Air friction effect + gravity

Frictional force proportional to v^2 with mg.

Here the constant $\underline{b_2}$, has the units of Kg/m and the constant $1/b_2 = L$ has the units of m .

The general solution to this problem is as follows.

$$F = mg - \underline{b_2} v^2 = ma \qquad L = \frac{1}{b_2}$$

$$\Downarrow$$

$$a = g - \frac{\underline{b_2}}{m} v^2 = g - b_2 v^2 = g - \frac{v^2}{L}$$

$$\Downarrow$$

$$\frac{dv}{dt} = g - b_2 v^2$$

$$\Downarrow$$

$$\frac{dv}{g - b_2 v^2} = dt$$

$$\Downarrow$$

$$\int_0^v \frac{dv}{g - b_2 v^2} = \int_0^t dt \qquad \int_0^v \frac{dv}{\frac{g}{b_2} - v^2} = b_2 \int_0^t dt$$

$$\Downarrow$$

$$v_t^{\,2} = \frac{g}{b_2} \qquad L = \frac{1}{b_2}$$

$$\Downarrow$$

$$\int_0^v \frac{dv}{v_t^{\,2} - v^2} = \frac{1}{L} \int_0^t dt$$

$$\Downarrow$$

$$\frac{1}{2v_t} \ln\left[\frac{v_t + v}{v_t - v}\right] = \frac{t}{L}$$

$$\Downarrow$$

$$\ln\left[\frac{v_t + v}{v_t - v}\right] = t\frac{2v_t}{L} = 2\frac{t}{\tau} \qquad \frac{v_t}{L} = \frac{1}{\tau}$$

$$\Downarrow$$

$$\left[\frac{v_t + v}{v_t - v}\right] = e^{\frac{2t}{\tau}} \qquad v = v_t \frac{[e^{\frac{2t}{\tau}} - 1]}{[e^{\frac{2t}{\tau}} + 1]} = v = v_t \frac{[e^{\frac{t}{\tau}} - e^{\frac{-t}{\tau}}]}{[e^{\frac{t}{\tau}} + e^{\frac{-t}{\tau}}]}$$

$$\Downarrow$$

$$\boxed{v = v_t \tanh\left(\frac{t}{\tau}\right)}$$

$$\frac{1}{v_t}\frac{dx}{dt} = \tanh\left(\frac{t}{\tau}\right)$$

$$\Downarrow$$

$$\frac{1}{v_t}\int_0^x dx = \int_0^t \tanh\left(\frac{t}{\tau}\right)dt$$

$$\Downarrow$$

$$\frac{x}{v_t} = \tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \Rightarrow \boxed{x = v_t \tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right]}$$