## Dynamics

### Rotational

<table>
<thead>
<tr>
<th>( \theta ) [radians]</th>
<th>( \omega = \frac{\Delta \theta}{\Delta t} ) [rad/sec]</th>
<th>( \alpha = \frac{\Delta \omega}{\Delta t} ) [rad/s²]</th>
</tr>
</thead>
</table>

\[ \omega = \frac{\Delta \theta}{\Delta t} \quad \text{[rad/sec]} \]

\[ \alpha = \frac{\Delta \omega}{\Delta t} \quad \text{[rad/s²]} \]

\[ I = \sum_{i} m_i r_i^2 \quad \text{[Kg m²]} \]

\[ \tau = r \perp F = rF \perp \quad \text{[Nm]} \]

\[ \tau = I \alpha \quad \text{[Nm]} \]

\[ K = \frac{1}{2} I \omega^2 \quad \text{[J]} \]

### Translation

<table>
<thead>
<tr>
<th>( x ) [m]</th>
<th>( v = \frac{\Delta x}{\Delta t} ) [m/s]</th>
<th>( a = \frac{\Delta v}{\Delta t} ) [m/s²]</th>
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\[ v = \frac{\Delta x}{\Delta t} \quad \text{[m/s]} \]

\[ a = \frac{\Delta v}{\Delta t} \quad \text{[m/s²]} \]

### Statics

**Equilibrium conditions**

\[ \sum_{i} \vec{F}_i = 0 \]

\[ \sum_{i} \vec{\tau}_i = 0 \]

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**Rolling without slipping**

\[ x = r \theta \quad v = \omega r \quad a = \alpha r \]
one object – circular (non-uniform !!) motion

\[ F_{\perp} = ma_{\perp} \]
\[ rF_{\perp} = r(ma_{\perp}) \]
\[ \tau = mr(a_{\perp}) \]
\[ \tau = mr^2 \alpha \]

Now add 2\textsuperscript{nd} mass \( m_1 \) and \( m_2 \) rigidly attached (“rigid body rotation”)

\[ \tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha \]
\[ \tau = [m_1 r_1^2 + m_2 r_2^2] \alpha \]

\[ \tau = [I] \alpha \]

I=moment of inertia

many masses

\[ I = \sum_{i}^{all} m_i r_i^2 \]
Objects tendency to resist changes in rotational velocity (like mass does for translation).

Two hoops with same mass

\[ I = mR^2 \]
\[ I' = m\left(\frac{R}{2}\right)^2 = \frac{1}{4}mR^2 \]

Harder to change its rotation speed!

Exert same torque, \( \tau \) on both.

\[ \tau = I\alpha \]

\[ \tau = (MR^2)\alpha_1 \]
\[ \alpha_1 = \frac{\tau}{(MR^2)} \]

\[ \tau = \frac{1}{4}(MR^2)\alpha_2 \]
\[ \alpha_2 = 4\frac{\tau}{(MR^2)} \]

4 times the angular acceleration 10-2
For Rolling, Moment of Inertia independent of length of cylinder.

Shape is important. A sphere will roll faster than a solid cylinder and a solid cylinder faster than a hoop cylinder.

Ratios: 2/5 vs. 1/2 vs. 1
At MSLC – top floor ARC
\[ \tau = I \alpha \]

\[ -TR = I \alpha \]

\[ a = \alpha R \]

\[ T - mg = ma \]

\[ \frac{-I}{R} (\alpha) - mg = ma \]

\[ \frac{I}{R} a - mg = ma \]

\[ -mg = ma + a \left( \frac{I}{R^2} \right) \]

\[ -g = a \left( 1 + \frac{I}{mR^2} \right) \]

\[ a = \frac{-g}{\left[ 1 + \frac{I}{mR^2} \right]} \]

\[ \text{or} \]

\[ I = mR^2 \left[ \frac{g}{|a|} - 1 \right] \]

See next page
\[ \Delta y = -\frac{a}{2} t^2 \]

\[ a = -\frac{2D}{t^2} = -\frac{2 \times 0.9}{3^2} = 0.2 \text{ m/s}^2 \]

\[ I = mR^2 \left[ \frac{g}{|a|} - 1 \right] \]

\[ I = 0.2 \times (0.115)^2 \left[ \frac{9.8}{0.2} - 1 \right] = 0.127 \text{ kg m}^2 \]
Rotation Kinetic Energy

\[
K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2 = \frac{1}{2} (m r^2) \omega^2
\]

For a Solid Disk, add all pieces:

\[
I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots
\]

Trans. KE  |  Rot. KE  |  Pot. E
---|---|---
\[E = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + mgh\]
\[\therefore E = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{R^2} + mgh\]
\[E = \frac{1}{2} (m + \frac{I}{R^2}) v^2 + mgh\]

\[E_f = E_i\]

\[\frac{1}{2} \left( m + \frac{I}{R^2} \right) v_i^2 + mgh_i = \frac{1}{2} \left( m + \frac{I}{R^2} \right) v_f^2 + mgh_f\]

\[\therefore \frac{1}{2} v_f^2 \left[ m + \frac{I}{R^2} \right] = \frac{1}{2} v_i^2 \left[ m + \frac{I}{R^2} \right] + mg(h_i - h_f)\]

\[v_f^2 = v_i^2 + \frac{2mg}{\left[ m + \frac{I}{R^2} \right]} (h_i - h_f)\]

or \[v_f^2 = v_i^2 + \frac{2g}{\left[ 1 + \frac{I}{mR^2} \right]} (h_i - h_f)\]
Rolling (without slipping) Down Inclined Plane

Translation: \((\sum \vec{F} = \vec{ma})\), where \(a\)=net acceleration down the plane

- \(\perp\) to plane \(N - mg \cos \theta = 0\)
- \(\parallel\) to plane \(+F_f - mg \sin \theta = ma\)

\(F_f = ma + mg \sin \theta\)

Note: only friction causes rotation/rolling !!!

Note: static friction=no sliding!!

R.H.R.

Note: Right Hand Rule applies to rolling CCW. The + direction can be “down hill” as shown above.
+ direction (CCW)

\[ F_f = m\alpha + mg \sin \theta \]

No slip \[ a = \alpha R \]

\[-(ma + mg \sin \theta)R = I\alpha = I \frac{a}{R}\]

\[-g \sin \theta = a \left( \frac{I}{mR^2} + 1 \right)\]

\[ a = \frac{-g \sin \theta}{(I/mR^2 + 1)} \]

\[ +F_f - mg \sin \theta = ma \]

\[ \tau = I\alpha \]

\[ -F_f R = I\alpha \]

only friction gives torque!!!
Reminder: The zero point of Potential Energy, $U$, can be defined at coordinate. We choose $U = 0$ at $y_i = 0$. Thus, for this problem:

\[ E_i = 0 \]

then \[
E_f = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2 + mg(-h)
\]

As usual, we assume no slipping: $v = \omega R$

\[ v = \sqrt{\frac{2gh}{1 + \frac{I}{mR^2}}} \]

larger $I$, will have smaller $v$. 

\[ 0 = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2 - mgh \]

\[ 0 = \frac{1}{2} mv^2 + \frac{1}{2} I \frac{v^2}{R^2} - mgh \]

\[ 0 = v^2 [\frac{m}{2} + \frac{I}{2R^2}] - mgh \]
<table>
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<th>Rotational</th>
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<tbody>
<tr>
<td>$p = mv$</td>
<td>$L = r_p = mvr$</td>
</tr>
<tr>
<td>$F = \frac{\Delta p}{\Delta t}$</td>
<td>$\tau = \frac{\Delta L}{\Delta t}$</td>
</tr>
<tr>
<td>$F = 0 \Rightarrow \Delta p = 0$</td>
<td>$\tau = 0 \Rightarrow \Delta L = 0$</td>
</tr>
<tr>
<td>$p_i = p_f$</td>
<td>$L_i = L_f$</td>
</tr>
<tr>
<td>p conserved</td>
<td>L conserved</td>
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</table>
The line of motion is shown as a dashed line with the axis marked. The angular momentum $\vec{L}$ is represented by a red arrow. The magnitude of $\vec{L}$ is given by the product of mass $m$, velocity $v$, and a perpendicular distance $r_{\perp}$.

\[ |\vec{L}| = mvr_{\perp} \]

\[ |\vec{L}| = pr_{\perp} \]
One object in rotation about axis

Angular momentum

\[ L = rp \quad v = \omega r \]
\[ L = rmv = mr^2 \omega \]
\[ L = \omega I \]
\[ I = mr^2 \]

Group of objects

\[ L_{\text{tot}} = L_1 + L_2 + L_3 + \ldots \]

For group or for solid (rigid body)

\[ L_{\text{tot}} = \omega I \]
\[ I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots \]

Add up over solid
Introduce force, $f$, to change rotational motion

\[ F = \frac{\Delta (mv)}{\Delta t} \]

$r = \text{constant}$

\[ rF = r \frac{\Delta (mv)}{\Delta t} = \frac{\Delta (rmv)}{\Delta t} \]

\[ rF = \frac{\Delta L}{\Delta t} \]

\[ \tau = \frac{\Delta L}{\Delta t} \]

\[ \tau = I \frac{\Delta \omega}{\Delta t} \]

\[ \tau = I \alpha \]

\[ \alpha = \frac{\Delta \omega}{\Delta t} \]
Conservation of angular momentum

\[ \tau = \frac{\Delta L}{\Delta t} \implies \Delta L = 0 \implies L \text{ conserved} \]

\[ L_i = L_f \]

Example: two parts of system exchange ang. mom.

\[ L = \ell_1 + \ell_2 \]

\[ \tau = 0 \implies \Delta L = 0 \]

but \[ \Delta L = 0 = \Delta \ell_1 + \Delta \ell_2 \]

\[ \therefore \Delta \ell_1 = -\Delta \ell_2 \]

suppose \[ \ell_1 = I_1 \omega_1 \quad \ell_2 = I_2 \omega_2 \]

then \[ \Delta (I_1 \omega_1) = -\Delta (I_2 \omega_2) \]

If \( I_1 \) and \( I_2 \) constant

\[ I_1 \Delta \omega_1 = -I_2 \Delta \omega_2 \]
2 counter-rotating blades

Tail rotator perpendicular (prevent rotation)

Tail rotator pushes forward

Typical helicopter

Main Rotor path

www.sikorsky.com
Sikorsky X2 (288 mph)

new helicopter
Example: internal forces rearrange mass distribution

\[ I_i = 2mr_i^2 \]
\[ L_i = \omega_i I_i \]
\[ I_f = 2mr_f^2 \]
\[ L_f = \omega_f I_f \]

\[ L_i = L_f \quad (\tau_{\text{tot-external}}=0) \]
\[ \omega_i I_i = \omega_f I_f \]
\[ \omega_f = \omega_i \frac{I_i}{I_f} = \omega_i \frac{2mr_i^2}{2mr_f^2} \]
\[ \omega_f = \omega_i \left(\frac{r_i}{r_f}\right)^2 \]

Rotating Chair
Mass close to axis - faster rotation.

Angular momentum conservation.
\[ L_i = L_f \quad \omega_i I_i = \omega_f I_f \quad \omega_f = \omega_i \frac{I_i}{I_f} \]

Shorter day = faster \( \omega \)
So earth's moment of inertia decreased!!

Quake Moves Japan Closer to U.S. and Alters Earth's Spin

By KENNETH CHANG

The magnitude-8.9 earthquake that struck northern Japan on Friday not only violently shook the ground and generated a devastating tsunami, it also moved the coastline and changed the balance of the planet.

Global positioning stations closest to the epicenter jumped eastward by up to 13 feet.

Japan is “wider than it was before,” said Ross Stein, a geophysicist at the United States Geological Survey.

Meanwhile, NASA scientists calculated that the redistribution of mass by the earthquake might have shortened the day by a couple of millionths of a second and tilted the Earth’s axis slightly.

Not all of Japan jumped 13 feet closer to the United States, said Kenneth W. Hudnut, a geophysicist with the United States Geological Survey. The shifts occurred mostly in the area closest to the epicenter, and stations farther away reported much less movement.

That part of Asia, to the surprise of many who look at the geological map, sits on the North American tectonic plate, which wraps up and around the Pacific plate and extends a tentacle southward that part of Japan sits atop. The Pacific plate is moving about 3.5 inches a year in a west-northwest direction, and in that collision — what geologists call a subduction zone — the Pacific plate dives under the North American plate.

Most of the time, the two tectonic plates are stuck together, and the North American plate is squeezed, much like a playing card held between the thumb and forefinger.

As the fingers squeeze the card, it buckles upward until the card pops free.

In the same way, the North American plate buckles, and the eastern part of Japan is slowly pushed to the west. But when the earthquake, which occurred offshore, released the tension, the land jumped back to the east.

As it unbuckled, a 250-mile-long coastal section of Japan dropped in altitude by two feet, which allowed the tsunami to travel farther and faster onto land, Dr. Stein said.

On a larger scale, the unbuckling and shifting moved the planet’s mass, on average, closer to its center, and just as a figure skater who spins faster when drawing the arms closer, the Earth’s rotation speeds up. Richard S. Gross, a scientist at NASA’s Jet Propulsion Laboratory, calculated that the length of the day was shortened by 1.8 millionths of a second.

Earlier great earthquakes also changed the axis and shortened the day. The magnitude-8.8 earthquake in Chile last year shortened the day by 1.26 millionths of a second and moved the axis by about three inches, while the Sumatra earthquake in 2004 shortened the day by 6.8 millionths of a second, Dr. Gross said.

Such changes are not unusual, and even without earthquakes, changes in ocean currents and atmospheric conditions usually have even greater effects. “The Earth is always wobbling, and the length of the day is always changing,” Dr. Gross said.

What is perhaps most surprising about the Japan earthquake is how misleading history can be. In the past 300 years, no earthquake nearly that large — nothing larger than magnitude-eight — had struck in the Japan subduction zone. That, in turn, led to assumptions about how large a tsunami might strike the coast.

“It did them a giant disservice,” said Dr. Stein of the geological survey. That is not the first time that the earthquake potential of a fault has been underestimated. Most geophysicists did not think the Sumatra fault could generate a magnitude-9.1 earthquake, and a magnitude-7.3 earthquake in Landers, Calif., in 1992 also caught earthquake experts by surprise.

“Perhaps the message is we should re-evaluate the occurrence of superlarge earthquakes on any fault,” Dr. Stein said.

NY Times 3/15/2011
Equilibrium

Does not translate

\[ \Sigma F = ma = 0 \]

no acceleration

Does not rotate

\[ \Sigma \tau = \Sigma F \cdot r = \frac{\Delta \theta}{\Delta t} = 0 \]

\[ \therefore \Sigma \tau = \Sigma F \cdot r = 0 \]

many forces
many torques
Footbridge - 2 supports - plank - man on plank

**Method 1**

\[ \sum F = 0 \]
\[ -Mg - mg + S_1 + S_2 = 0 \]

\[ \sum \tau = 0 \text{ (axis 1)} \]
\[ -Mg \cdot \frac{L}{4} - mg \cdot \frac{3L}{4} + S_2 \cdot L = 0 \]

\[ \Rightarrow S_2 = \frac{mg}{L} - \frac{3mg}{4} \]

\[ \sum \tau = 0 \text{ (axis 2)} \]
\[ -Mg \cdot \frac{L}{4} - mg \cdot \frac{L}{2} - S_1 \cdot L = 0 \]

\[ \Rightarrow S_1 = \frac{mg}{4} + \frac{mg}{2} \]

... solve in 2 steps
Door - what are forces why upper hinge?

\[ \Sigma F_y = 0 \]
\[ v + v' - w = 0 \]
\[ \Sigma F_x = 0 \]
\[ -H + H' = 0 \]

Assuming \( v = v' \)

\[ \Sigma \zeta = 0 \]
\[ 2v - w = 0 \]
\[ v = \frac{w}{2} \]

\[ = -w \cdot 2 + H \cdot 6 = \frac{2w}{3} \]

\[ \Rightarrow H = \frac{w}{3} \]
F\text{loor} \ (friction)

Floor (friction)

\[ F_{Nw} = \frac{2}{3} \text{mg} \cos \theta \]

\[ \leq F_x = m a_x = 0 \]

\[ + F_{f} - N_{w} = 0 \implies F_{f} = N_{w} \]

\[ \leq F_y = m a_y = 0 \]

\[ + N_{F} - \text{mg} = 0 \implies N_{F} = \text{mg} \]

\[ \leq \theta = \frac{\alpha_{1} \beta_{2}}{\Delta} = 0 \]

\[ + N_{w} \gamma_{n} - \text{mg} \gamma_{w} + N_{F} \gamma_{F} + F_{f} \gamma_{F} = 0 \]

\[ N_{w} \sin \theta - \text{mg} \frac{2\theta}{3} \cos \theta = 0 \]

\[ N_{w} = \frac{2}{3} \text{mg} \cos \theta \frac{\sin \theta}{\tan \theta} = \frac{2}{3} \text{mg} \frac{\sin \theta}{\tan \theta} = F_{f} \]