

## 2-D $\vec{a} = \text{constant}$ motion $\vec{a} \Leftrightarrow (a_x, a_y)$ constant

$$\mathbf{t} = \mathbf{0} \quad \vec{v}_0 \Leftrightarrow (v_{0x}, v_{0y}) \quad \vec{r}_0 \Leftrightarrow (x_0, y_0)$$

Most General Form (**actually rarely necessary**)

- $v_x = v_{0x} + a_x t$
- $x = x_0 + v_{0x} t + a_x \frac{t^2}{2}$
- $\Delta x = \bar{v}_x t = \frac{(v_{0x} + v_x)}{2} t$
- $v_x^2 - v_{0x}^2 = 2a_x (x - x_0)$
- $v_y = v_{0y} + a_y t$
- $y = y_0 + v_{0y} t + a_y \frac{t^2}{2}$
- $\Delta y = \bar{v}_y t = \frac{(v_{0y} + v_y)}{2} t$
- $v_y^2 - v_{0y}^2 = 2a_y (y - y_0)$

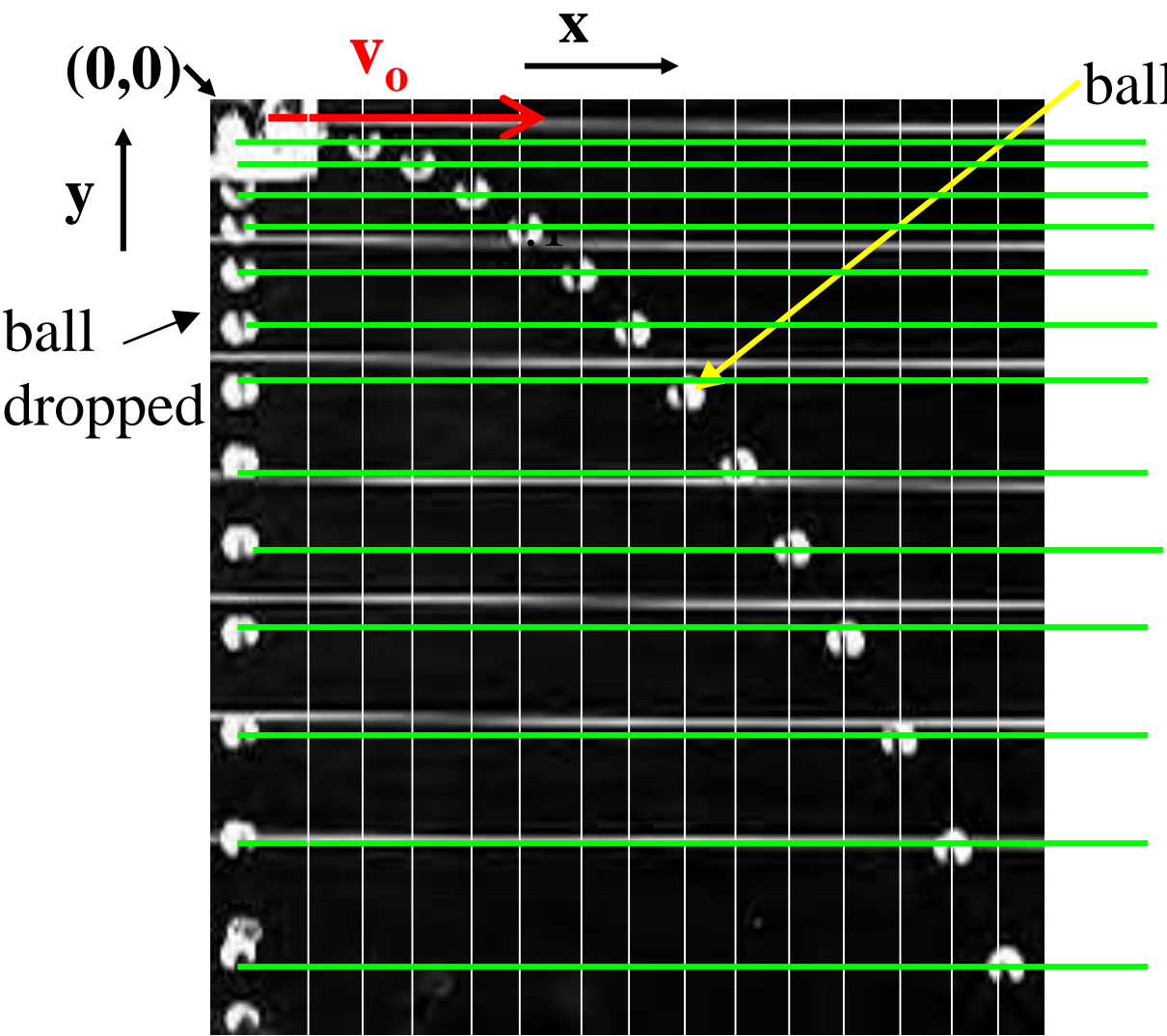
**For special case (all objects projected near the surface of the earth)**

$$\vec{\mathbf{a}} \Leftrightarrow \begin{array}{l} \mathbf{a}_x = 0 \\ \mathbf{a}_y = -g \end{array} \quad \begin{array}{c} \downarrow g \\ \uparrow +y \end{array}$$

**$\mathbf{t} = 0 \quad \vec{\mathbf{r}}_0 \Leftrightarrow (\mathbf{r}_{0x}, \mathbf{r}_{0y}) \quad \vec{\mathbf{v}}_0 \Leftrightarrow (\mathbf{v}_{0x}, \mathbf{v}_{0y})$  Initial conditions**

- $\mathbf{v}_x = \mathbf{v}_{0x}$
- $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_{0x} \mathbf{t}$
- $\mathbf{v}_y = \mathbf{v}_{0y} - g \mathbf{t}$
- $y = y_0 + \mathbf{v}_{0y} \mathbf{t} - g \frac{\mathbf{t}^2}{2}$
- $(y - y_0) = \frac{(\mathbf{v}_{0y} + \mathbf{v}_y)}{2} \mathbf{t} = \bar{\mathbf{v}}_y \mathbf{t}$
- $\mathbf{v}_y^2 - \mathbf{v}_{0y}^2 = 2(-g)(y - y_0)$

**Begin with experiment**



- **increasing displacement in = time ( $\sim t^2$ )**
- **acceleration  $g$  in  $-y$  direction**

**y-dir  
projected ball motion  
=  
dropped ball motion**

$$v_y = -gt$$
$$y = -gt^2/2$$

**x-dir  
equal displacements in equal times  
constant velocity in x-dir**

$$v_x = v_{0x}$$
$$x = v_{0x} t$$

For special case (all objects projected near the surface of the earth)

$$\vec{\mathbf{a}} \Leftrightarrow \begin{matrix} \mathbf{a}_x = 0 \\ \mathbf{a}_y = -g \end{matrix} \quad \begin{matrix} \downarrow g \\ \uparrow +y \end{matrix}$$

$\mathbf{t} = 0 \quad \vec{\mathbf{r}}_0 \Leftrightarrow (\mathbf{r}_{0x}, \mathbf{r}_{0y}) \quad \vec{\mathbf{v}}_0 \Leftrightarrow (\mathbf{v}_{0x}, \mathbf{v}_{0y})$  **Initial conditions**

- $\mathbf{v}_x = \mathbf{v}_{0x}$
- $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_{0x} \mathbf{t}$
- $\mathbf{v}_y = \mathbf{v}_{0y} - g \mathbf{t}$
- $y = y_0 + \mathbf{v}_{0y} \mathbf{t} - g \frac{\mathbf{t}^2}{2}$
- $(y - y_0) = \frac{(\mathbf{v}_{0y} + \mathbf{v}_y)}{2} \mathbf{t} = \bar{\mathbf{v}}_y \mathbf{t}$
- $\mathbf{v}_y^2 - \mathbf{v}_{0y}^2 = 2(-g)(y - y_0)$

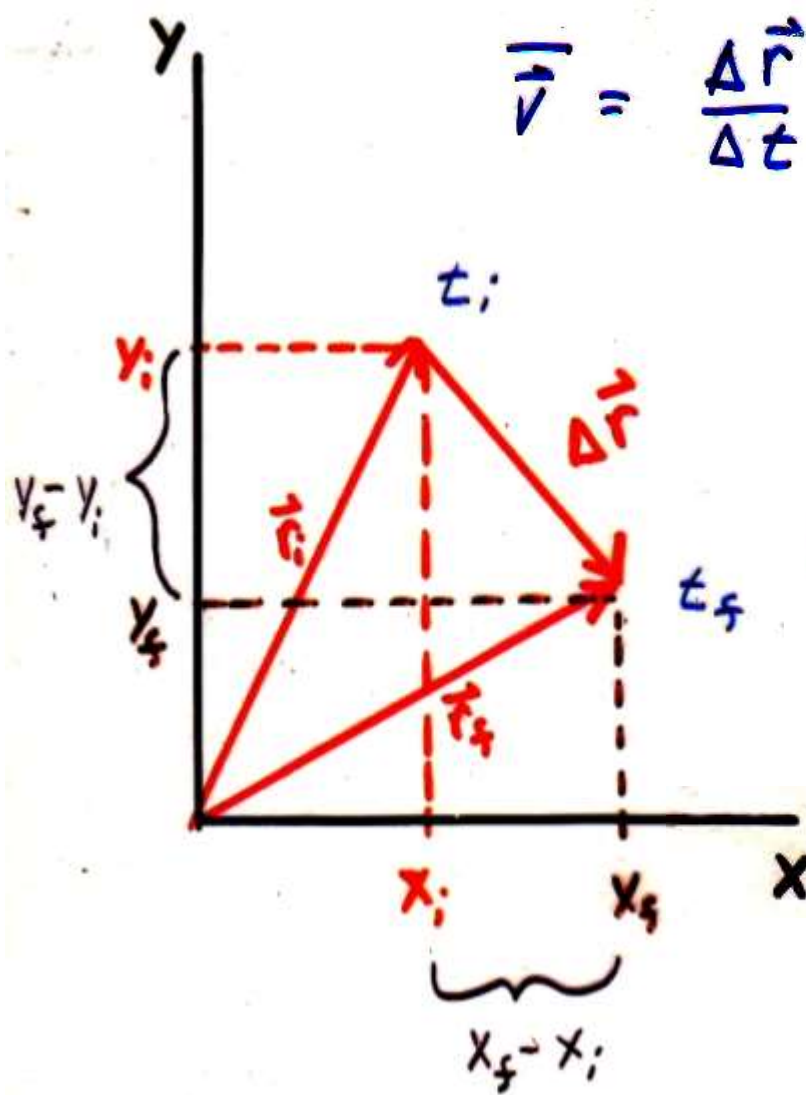
In general

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t} \quad \begin{matrix} \rightarrow \vec{v}_x = \frac{\Delta x}{\Delta t} \\ \downarrow \vec{v}_y = \frac{\Delta y}{\Delta t} \end{matrix}$$

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i$$

$$\Delta \vec{r} \text{ x comp} = x_f - x_i$$

$$\Delta \vec{r} \text{ y comp} = y_f - y_i$$



$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

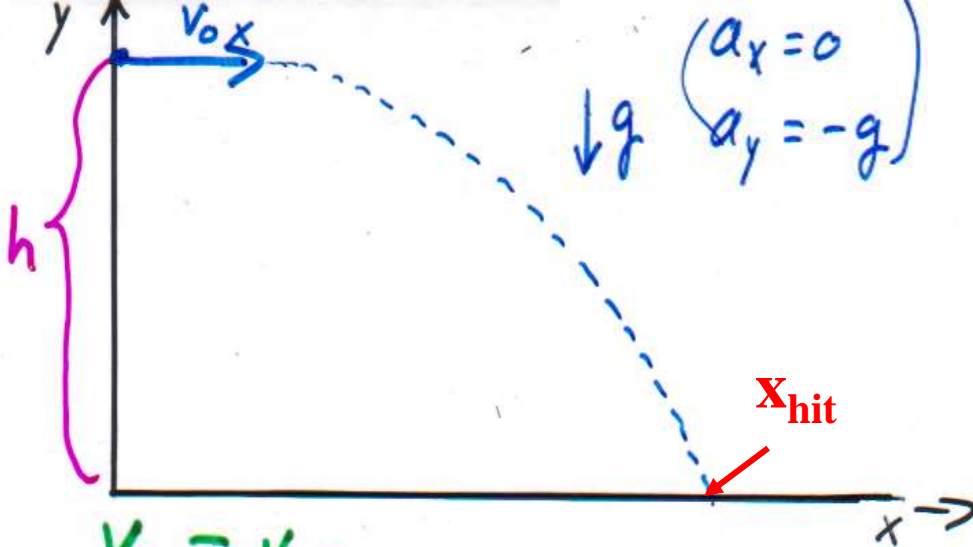
$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t}$$

Special Case  
again

$\vec{a}$  = constant  
magnitude  
& direction

• usually choose  
 $\vec{a}$  along y axis  
for convenience

## Object projected horizontally



Use  $x_{hit}$  and  $h$  to find  $v_o$

$$x_{hit} = v_{ox} \sqrt{\frac{2h}{g}}$$

$$v_{ox} = x_{hit} \sqrt{\frac{g}{2h}}$$

$$v_x = v_{ox}$$
$$x - x_0 = v_{ox} t \quad v_y = -gt$$
$$y = h - \frac{gt^2}{2}$$

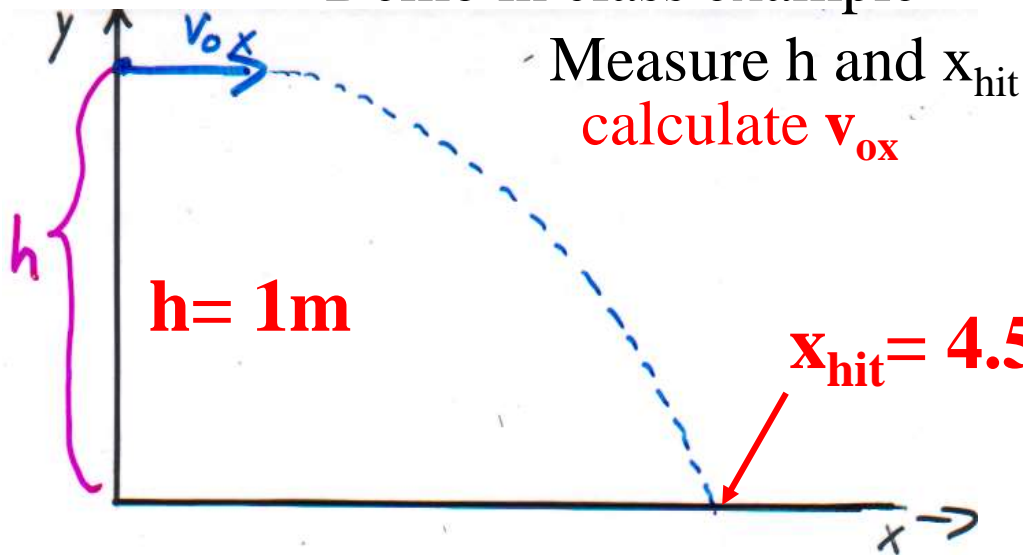
①  $t_{hit}$  ground & where  $(x_{hit}, y_{hit})$

$$y = 0 = h - \frac{gt^2}{2} \Rightarrow t_{hit} = \sqrt{\frac{2h}{g}}$$

$$x = v_{ox} t = v_{ox} \sqrt{\frac{2h}{g}} = x_{hit} \quad ; \quad y_{hit} = 0$$

# Demo in class example

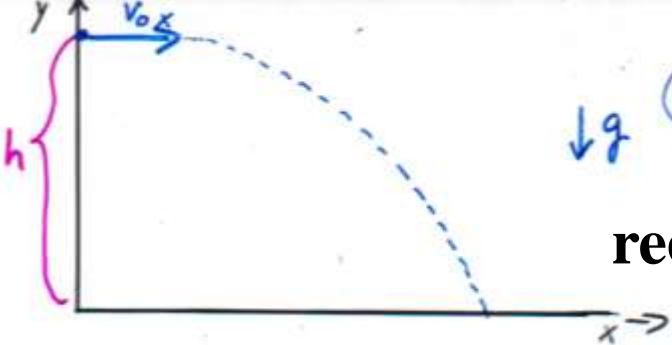
recall



$$t_{hit} = \pm \sqrt{\frac{2h}{g}}$$

$$v_{ox} t = v_{ox} \sqrt{\frac{2h}{g}} = x_{hit}$$

$$v_{ox} = \frac{x_{hit}}{t_{hit}} = x_{hit} \sqrt{\frac{g}{2h}} = 4.5 \text{ (m)} \sqrt{\frac{9.8 \text{ (m/s}^2\text{)}}{2 \cdot 1 \text{ (m)}}} = 10 \text{ m/s} = v_{ox}$$



recall

$\downarrow g \quad (a_x = 0, a_y = -g)$   
 $v_x = v_{0x}$   
 $x - x_0 = v_{0x} t$

calculate  $\vec{v}_{hit}$

$y = h - \frac{gt^2}{2}$

$v_y = -gt$

$y = 0 = h - \frac{gt^2}{2} \Rightarrow t_{hit} = \sqrt{\frac{2h}{g}}$

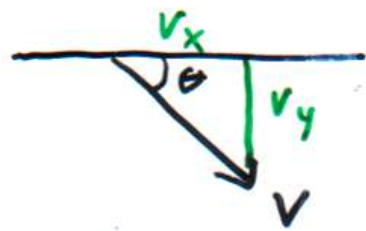
$\vec{v}_{hit} = ?$

$v_{y_{hit}} = -g t_{hit} = -g \sqrt{\frac{2h}{g}} = -\sqrt{2hg} = v_{y_{hit}}$

(or)  $v_y^2 - 0^2 = 2(-g)(0-h) \leftarrow v_y^2 - v_{0y}^2 = 2a_y(y - y_0)$

$v_{x_{hit}} = v_{0x}$

$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{0x}^2 + (2gh)}$

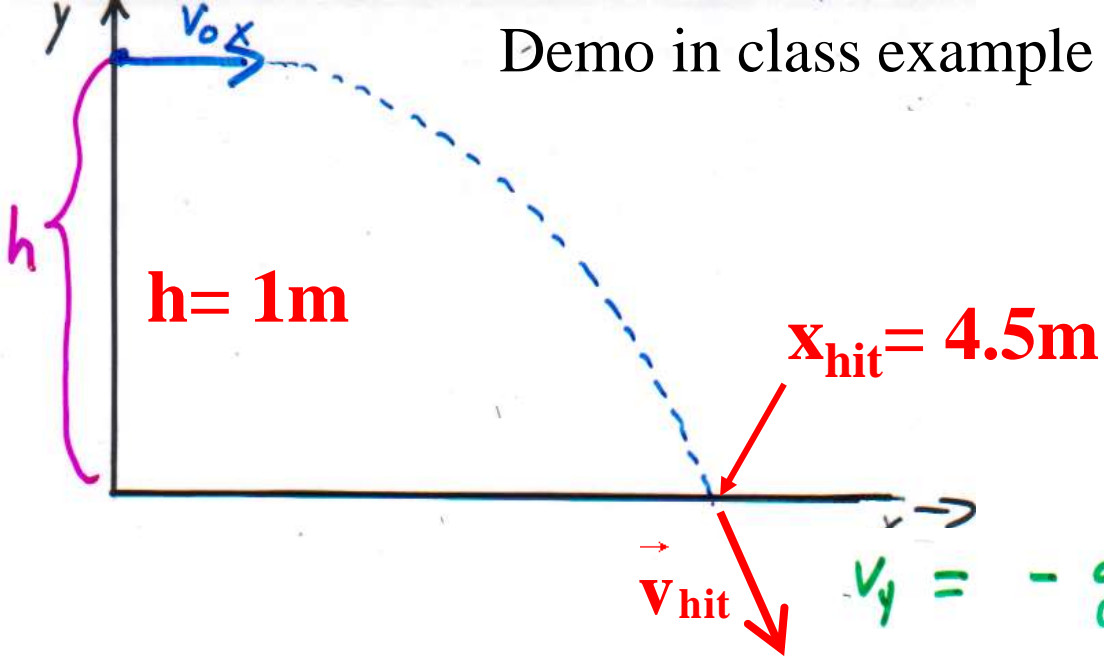


$\theta = \tan^{-1}\left(\frac{-\sqrt{2hg}}{v_{0x}}\right)$

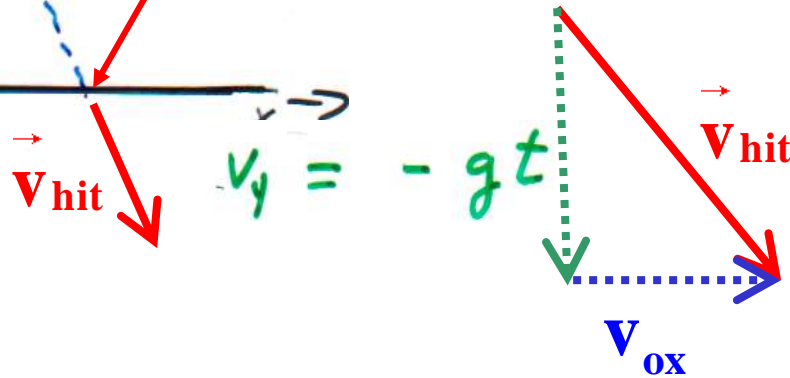


Demo in class example

recall



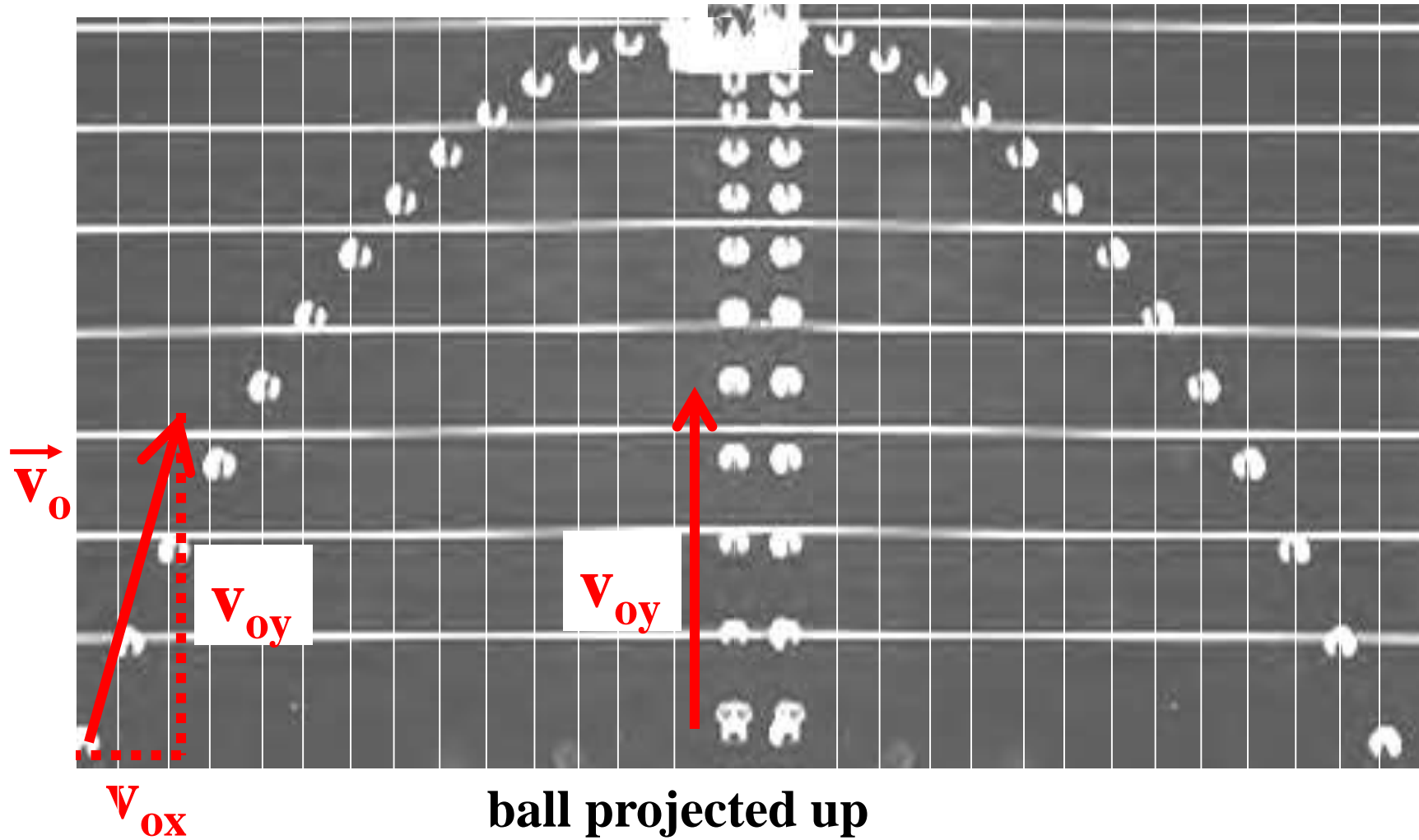
$$v_{ox} = \frac{x_{hit}}{\sqrt{\frac{2h}{g}}} = 10 \frac{\text{m}}{\text{s}}$$



$$v_y = -g t_{hit} = -\sqrt{2hg} = -\sqrt{19.6} \text{ ( m/s)} = -4.43 \text{ m/s}$$

$$|\vec{v}_{hit}| = \sqrt{v_x^2 + v_y^2} = \sqrt{10^2 + 4.43^2} = \sqrt{100 + 19.6} = 10.9 \text{ m/s}$$

# ball projected at angle



3-6

## ball projected up

ball  
projected  
at  
angle

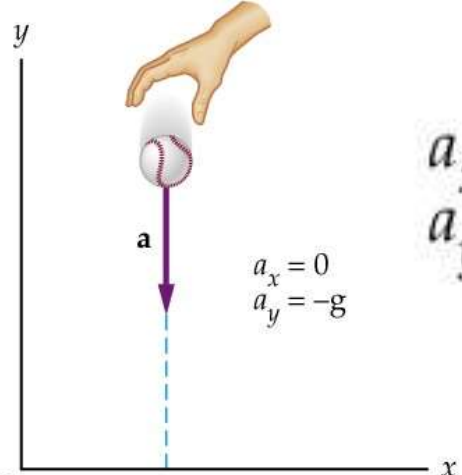
$$\begin{aligned} v_x &= v_{ox} \\ x &= v_{ox}t \end{aligned}$$

constant velocity  $v_{ox}$

$$\begin{aligned} y &= v_{oy}t - gt^2/2 \\ v_y &= v_{oy} - gt \end{aligned}$$

just like ball  
projected up  
with  $v_{oy}$

# Some Observations

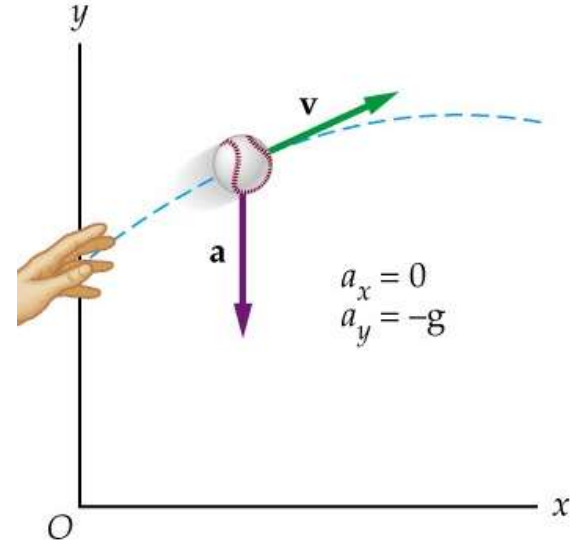


$$a_x = 0$$

$$a_y = -g$$

$$a_x = 0$$

$$a_y = -g$$

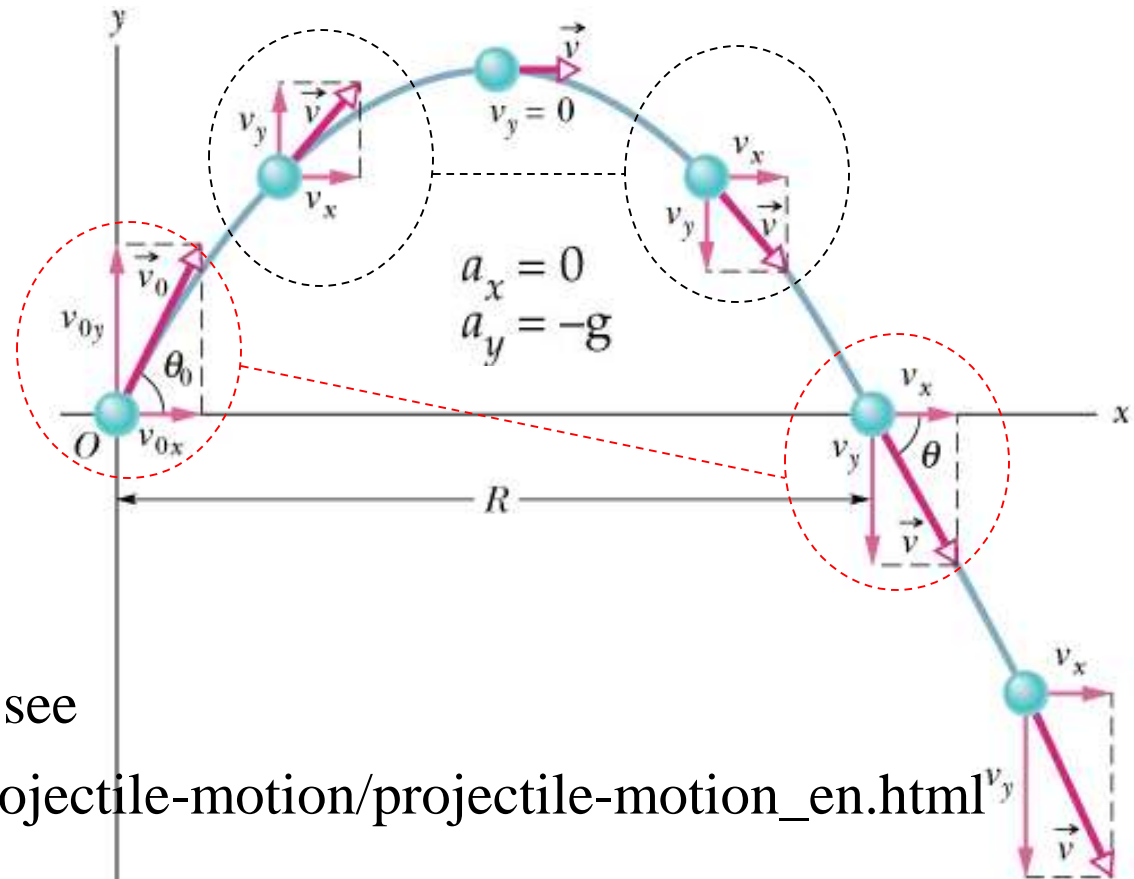


$$a_x = 0$$

$$a_y = -g$$

Note the following:

- $V_x$  and  $V_y$  independent
- $V_x$  remains constant
- The symmetries in  $V_y$   
The signs differ, but magnitudes are same
- Vector analysis  $\Rightarrow V$  is the slope of the curve.

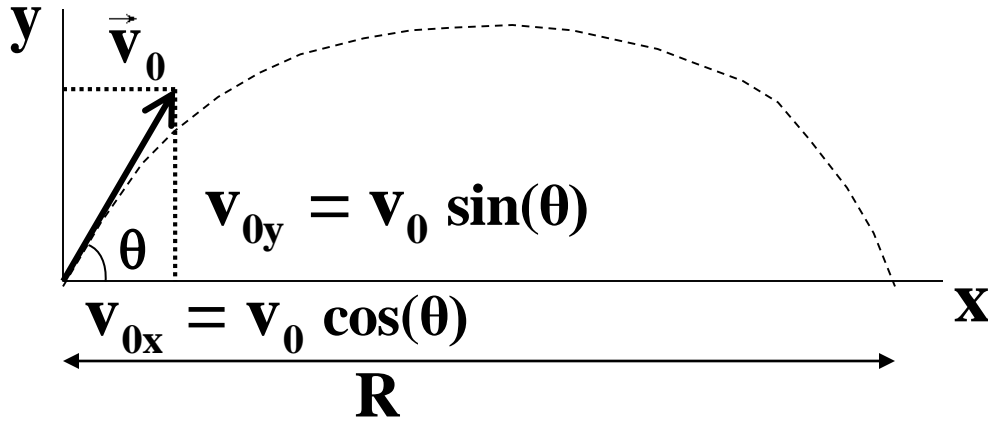


3-6a

Projectile motion simulation- see

[http://phet.colorado.edu/sims/projectile-motion/projectile-motion\\_en.html](http://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html)

# Projectile motion: Range



x-dir. const. vel.

$$v_x = v_{0x}$$

$$x = v_{0x} t$$

y-dir. const. acc.

$$v_y = v_{0y} - g t$$

$$y = v_{0y} t - g \frac{t^2}{2}$$

Range,  $R$ : when it hits the ground ( $y=0$ )

$$\therefore y = 0 = v_{0y} t - g \frac{t^2}{2} \quad \text{key!}$$

$$R = x(t_{\text{hit}}) = v_{0x} t_{\text{hit}} = v_{0x} \frac{2v_{0y}}{g}$$

$$R = \frac{2v_{0x} v_{0y}}{g}$$

$$0 = [t] \left[ v_{0y} - g \frac{t}{2} \right]$$

$$0 = \left[ v_{0y} - g \frac{t}{2} \right]$$

$t = 0$   
 $t=0$  shot from  $y=0$

$$t_{\text{hit}} = \frac{2v_{0y}}{g}$$

$t_{\text{hit}}$  hits the ground at  $y=0$

Range

so

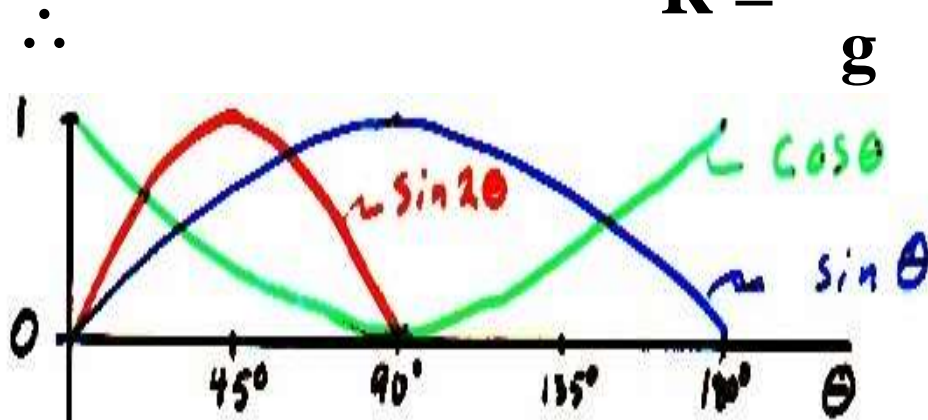
$$R = \frac{2v_{0x} v_{0y}}{g}$$

but

$$v_{0y} = v_0 \sin(\theta)$$

$$v_{0x} = v_0 \cos(\theta)$$

$$R = \frac{v_0^2}{g} 2 \sin(\theta) \cos(\theta) = \frac{v_0^2}{g} \sin(2\theta)$$



Note →  $\sin 2\theta$  symmetrical about  $\theta = 45^\circ$

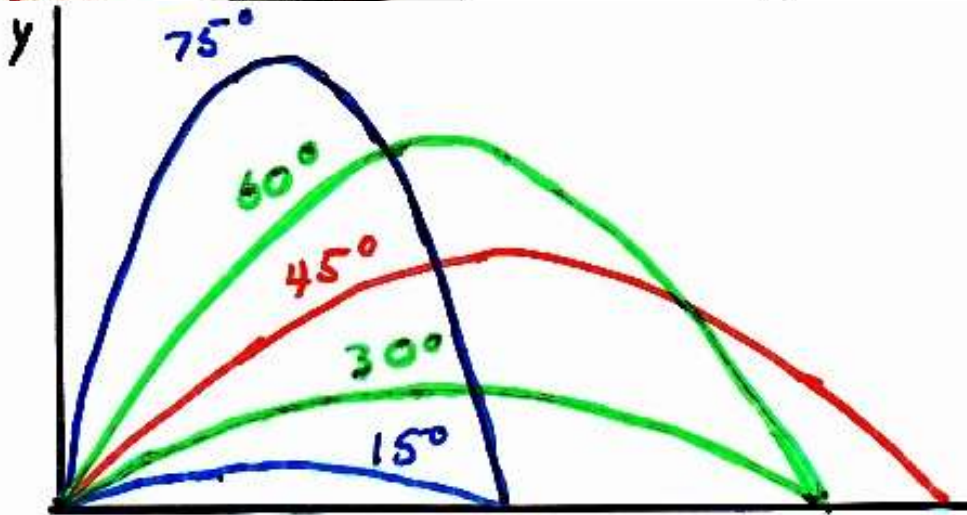
$$R = \frac{v_0^2}{g} \sin 2\theta$$

symmetrical  
about  $\theta = 45^\circ$

$45^\circ$  max. range

$45^\circ \pm \phi$  have the same R

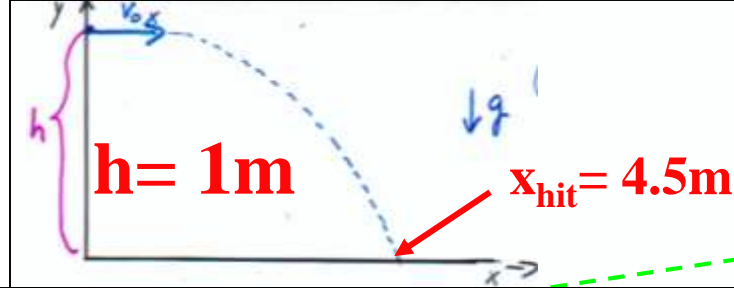
different  
max h  
& flight time



3-7a

Recall Demo in class

$$v_{ox} = x_{hit} \sqrt{\frac{g}{2h}} = 10 \frac{m}{s}$$

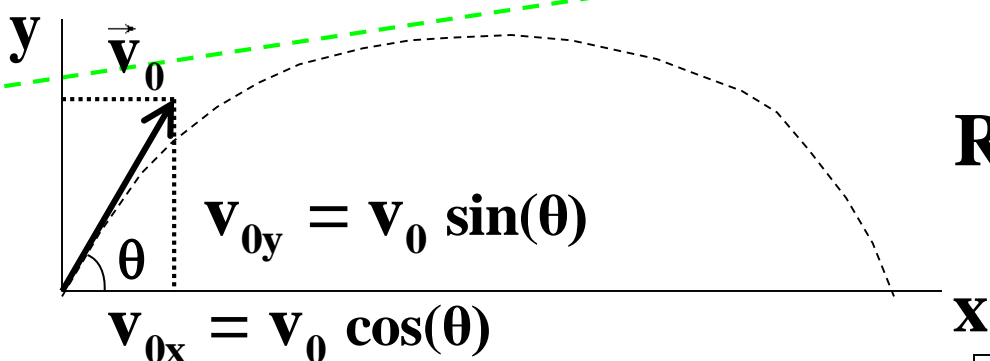


2<sup>nd</sup> class Demo

: range

same "gun"

=same  $v_0$

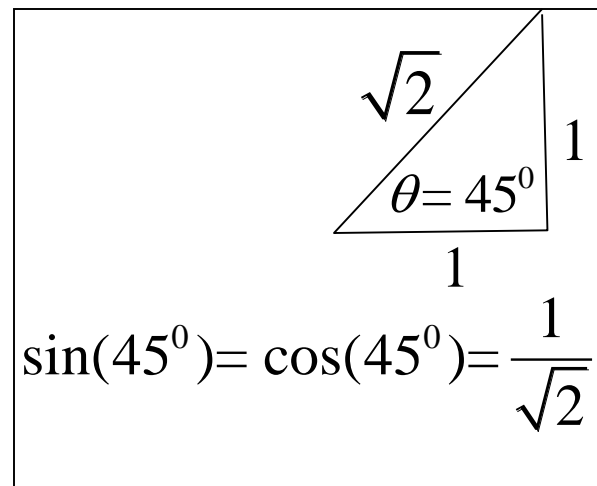


$$R = \frac{2 v_{ox} v_{oy}}{g}$$

Recall

$$\theta = 45^\circ \Rightarrow R_{max}$$

$$R_{max} = \frac{2 v_0^2}{g(\sqrt{2})^2} = \frac{v_0^2}{g}$$

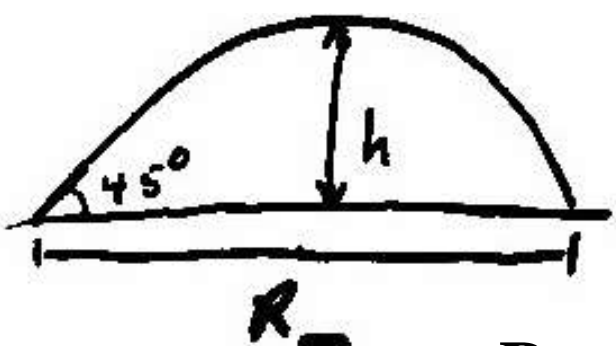


$$R_{max}(\text{class demo}) = \frac{10^2}{9.8} \left[ \frac{(m/s)^2}{m/s^2} \right] = 10.2m$$

**good agreement**

Scud missiles Iraq to Israel 1991 [~most of trajectory is that of a projectile]

Recall range problem max range  $\theta = 45^\circ$  clearly  $\theta = 45^\circ$  used to get max range



$$R_m = ? \quad v_0 = ?$$

$$h = ?$$

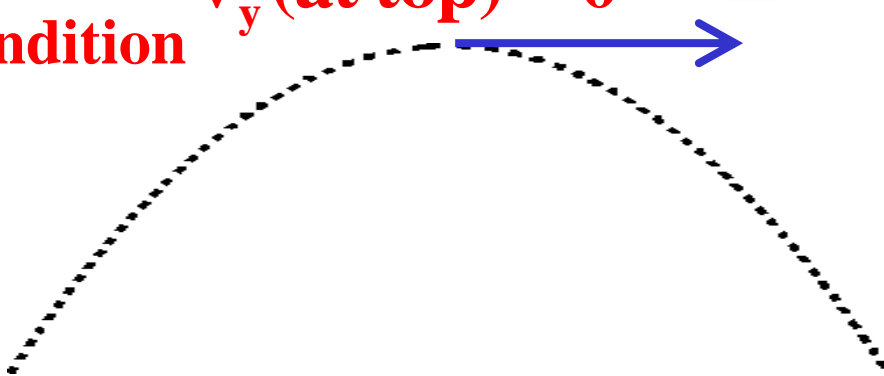
Time to get to max height (h)

$$R_{\max} = \frac{2 v_0^2}{g(\sqrt{2})^2} = \frac{v_0^2}{g} \quad \theta = 45^\circ \quad \sin(\theta) = \frac{1}{\sqrt{2}}$$

Time to get to max height (h)

crucial condition  $v_y$  (at top) = 0

$$v_y = v_{oy} - gt \quad \uparrow = 0$$



$$\Rightarrow t_{\text{top}} = \frac{v_{oy}}{g} = \frac{v_0 \sin(\theta)}{g}$$

$$\theta = 45^\circ \quad t_{\text{top}} = \frac{v_0}{g\sqrt{2}}$$

What is max height (h)

$$h = y(t_{\text{top}}) = v_{\text{oy}} t - \frac{g}{2} t^2$$

$$h = v_{\text{oy}} \frac{v_{\text{oy}}}{g} - \frac{g}{2} \left( \frac{v_{\text{oy}}}{g} \right)^2$$

$t_{\text{top}} = \frac{v_{\text{oy}}}{g}$
$\theta = 45^\circ$
$v_{\text{oy}} = \frac{v_0}{\sqrt{2}}$

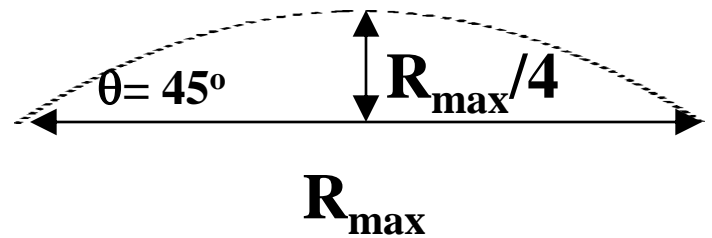
$$h = \frac{[v_{\text{oy}}]^2}{2g}$$

$$\theta = 45^\circ \quad h = \left[ \frac{v_0}{2} \right]^2 \frac{1}{2g}$$

$$h = \frac{v_0^2}{4g}$$

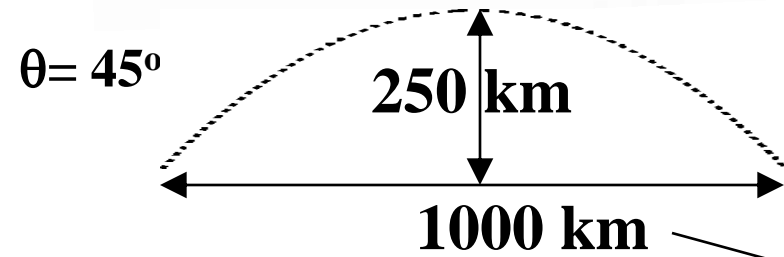
But recall
$R_{\text{max}} = \frac{v_0^2}{g}$

$$h = \frac{1}{4} R_{\text{max}}$$





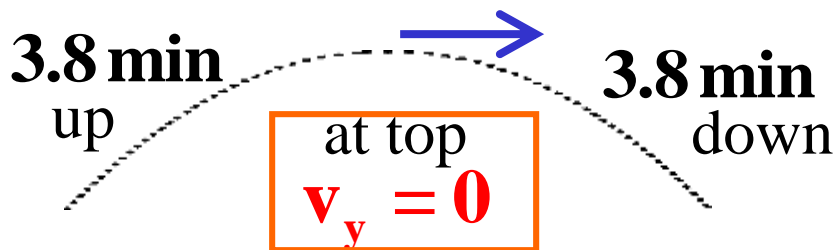
Scud missiles case 1991  $R_{\max} \approx 1000 \text{ km} = 10^6 \text{ m}$   $\approx 1000 \text{ km}$   
 $h = \frac{1}{4} R_m \approx \underline{250 \text{ km}} = 2.5(10)^5 \text{ m}$   $\theta = 45^\circ$  Israel Iraq



$$R_m = \frac{v_o^2}{g} \implies v_o = \sqrt{R_m g}$$

$$v_o = \sqrt{10^6 \text{ [m]} 9.8 \text{ [m/s}^2\text{]}} = \underline{3200 \text{ m/s}}$$

$$\theta = 45^\circ \quad t_{\text{top}} = \frac{v_o}{g\sqrt{2}} \quad t_{\text{top}} = \frac{3200 \text{ [m/s]}}{9.8 \text{ [m/s}^2\text{]}\sqrt{2}} = 229 \text{ s} = 3.8 \text{ min}$$

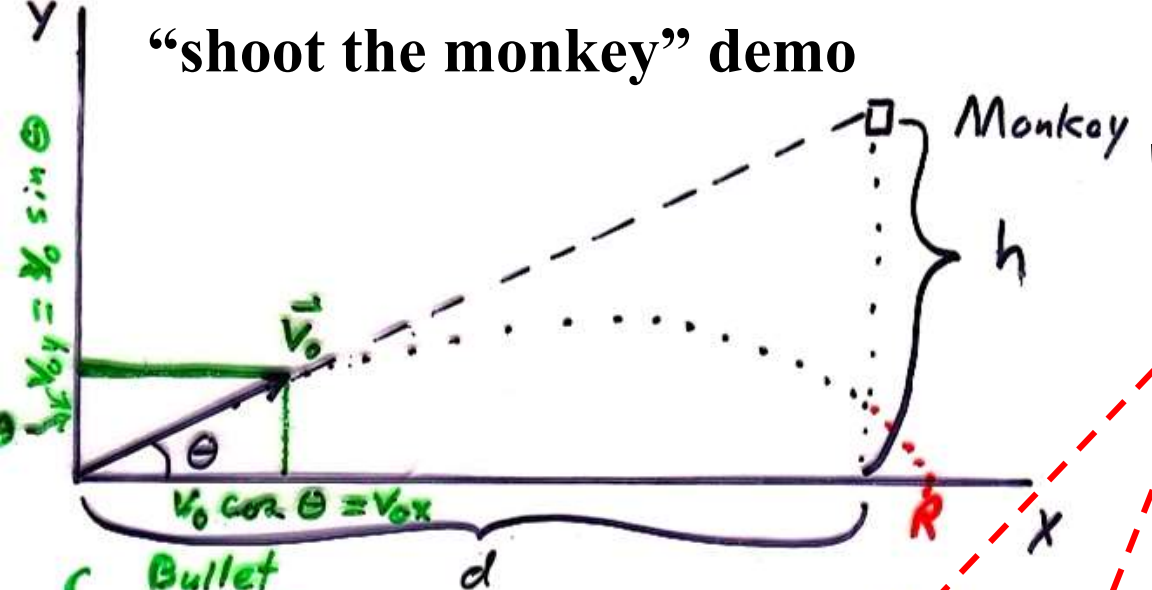


7.6 min total warning time

$$\theta = 45^\circ \quad v_x = v_{0x} = \frac{v_o}{\sqrt{2}} = \underline{2300 \text{ m/s}}$$

At top slowest  
– try to intercept at top

# "shoot the monkey" demo



Bullet

$$\begin{cases} x = v_{0x} t \\ y = v_{0y} t - \frac{g}{2} t^2 \end{cases}$$

Monkey

$$\begin{cases} y_m = h - \frac{g}{2} t^2 \\ x_m = d \end{cases}$$

To hit monkey

$x = x_m$   
 $y = y_m$  } at same time  $t$

$$\begin{aligned} x &= x_m \\ v_{0x} t &= d \\ \downarrow \\ t &= d/v_{0x} \end{aligned}$$

$$\begin{aligned} v_{0y} t - \frac{g}{2} t^2 &= h - \frac{g}{2} t^2 \\ \downarrow \\ v_{0y} t &= h \end{aligned}$$

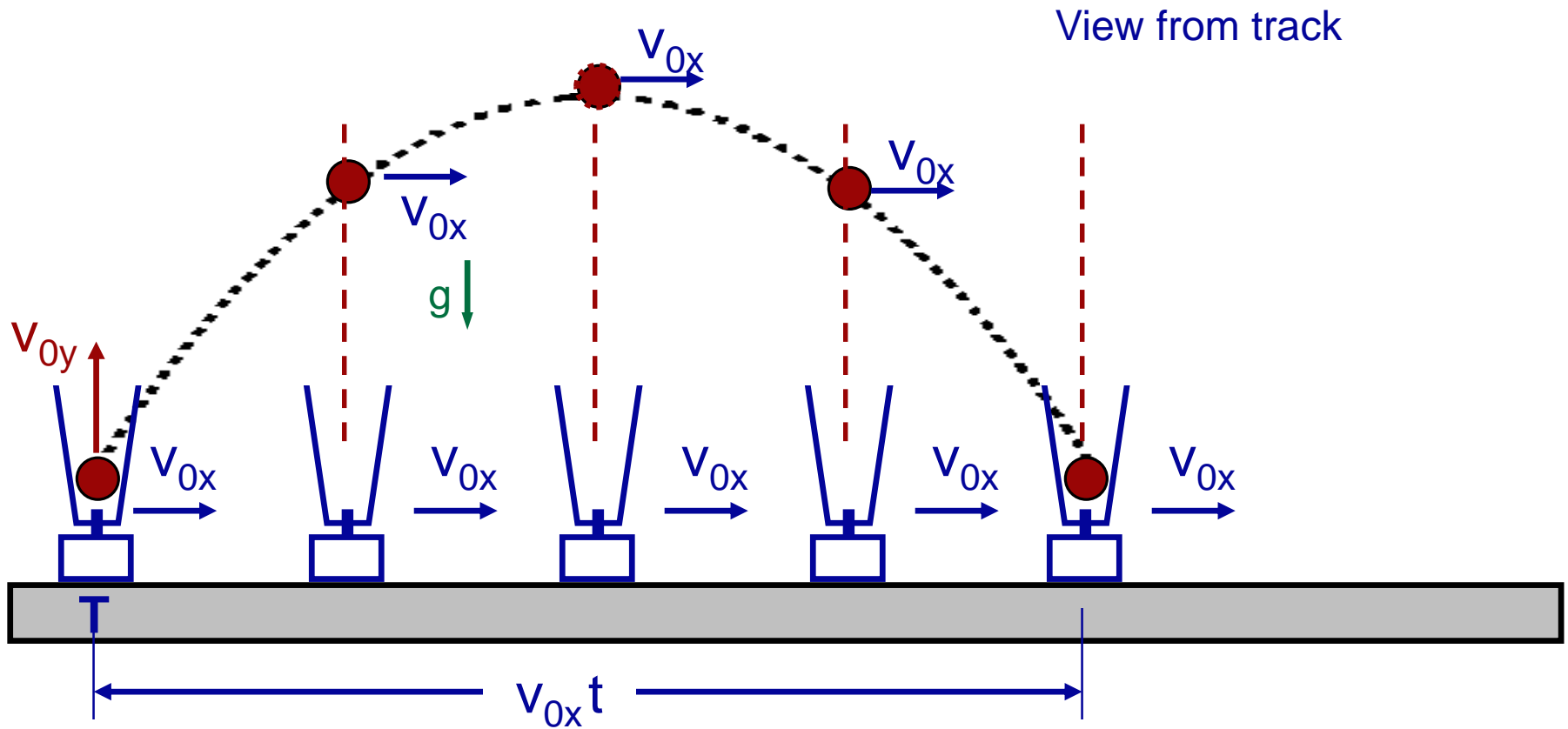
$$\frac{v_{0y} d}{v_{0x}} = h$$

$$\frac{v_0 \sin \theta d}{v_0 \cos \theta} = h$$

$$\boxed{\tan \theta = \frac{h}{d}}$$

Aim at monkey !!

# Howitzer Cart



Cart moves at constant velocity. As the cart passes point "T" a firing pin is triggered, shooting the ball straight up. Where will the ball land?

x - motion = constant velocity for both

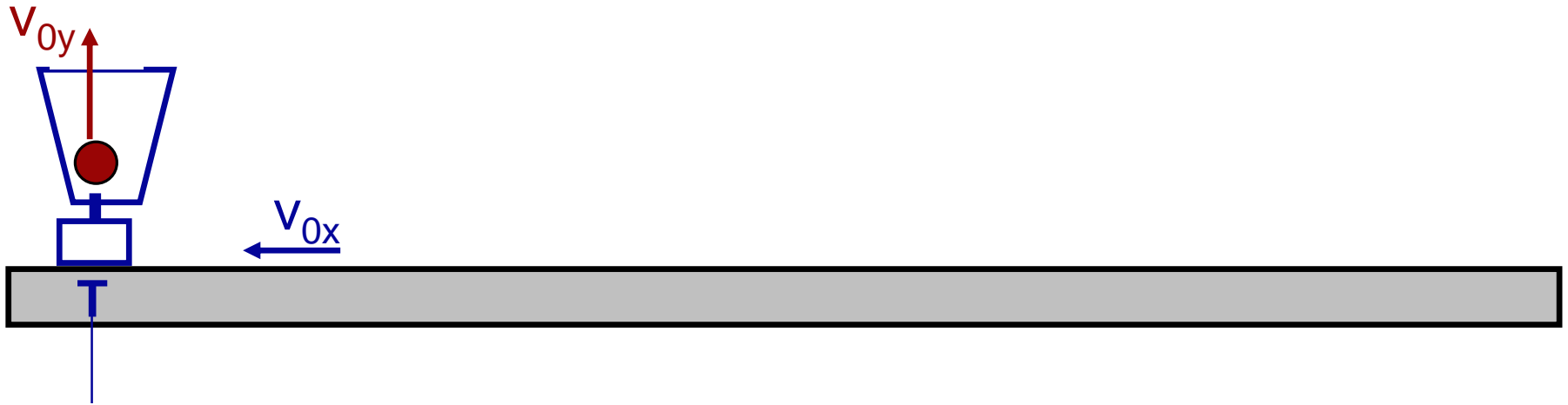
A. BACK IN CART

Good old reference frame video see  
<http://www.youtube.com/watch?v=Y75kEf8xLxI>

# Howitzer Cart

View from cart

x – motion = track moves at constant velocity

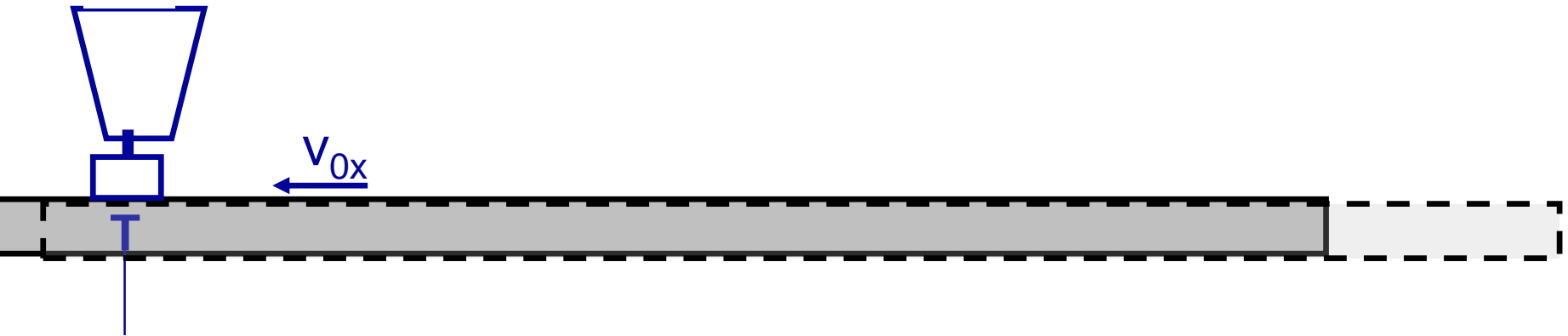


A. OBVIOUSLY BACK IN CART

# Howitzer Cart

View from cart

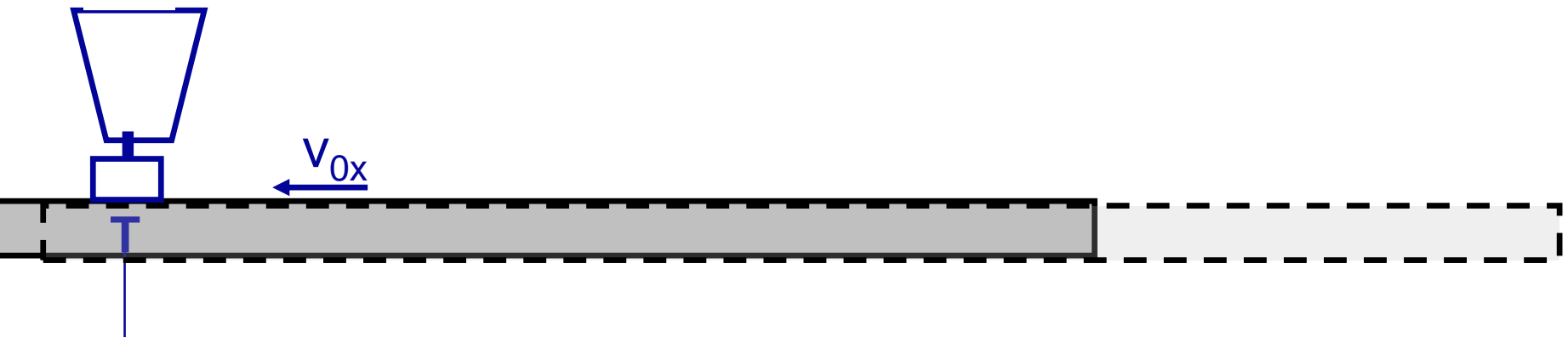
x – motion = track moves at constant velocity



A. OBVIOUSLY BACK IN CART

# Howitzer Cart

View from cart  
x – motion = track moves at constant velocity

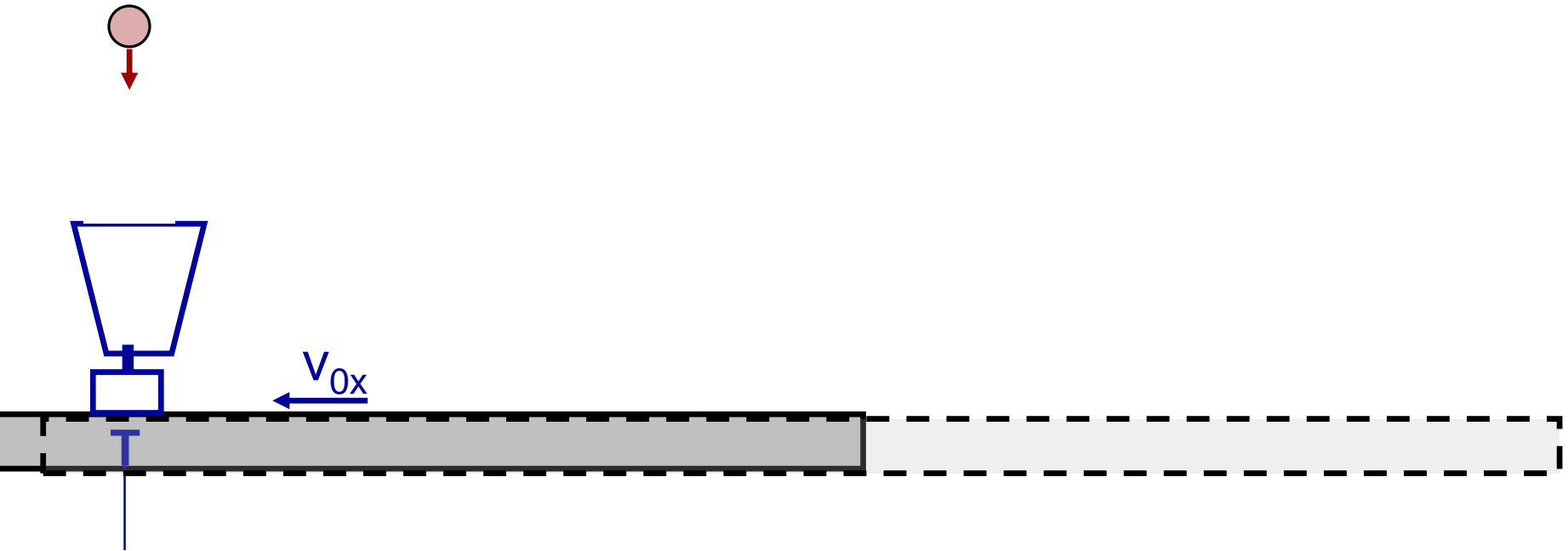


A. OBVIOUSLY BACK IN CART

# Howitzer Cart

View from cart

x – motion = track moves at constant velocity

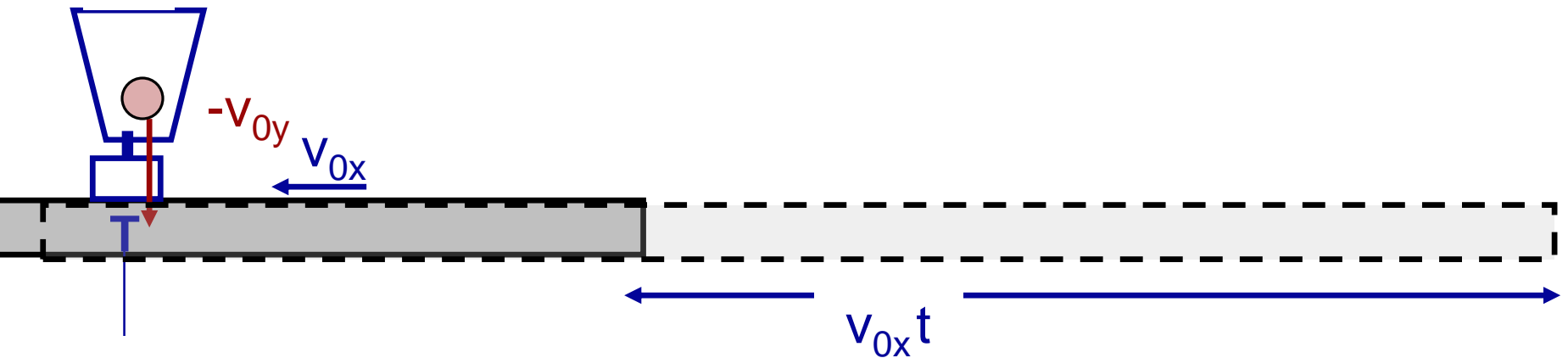


A. OBVIOUSLY BACK IN CART

# Howitzer Cart

View from cart

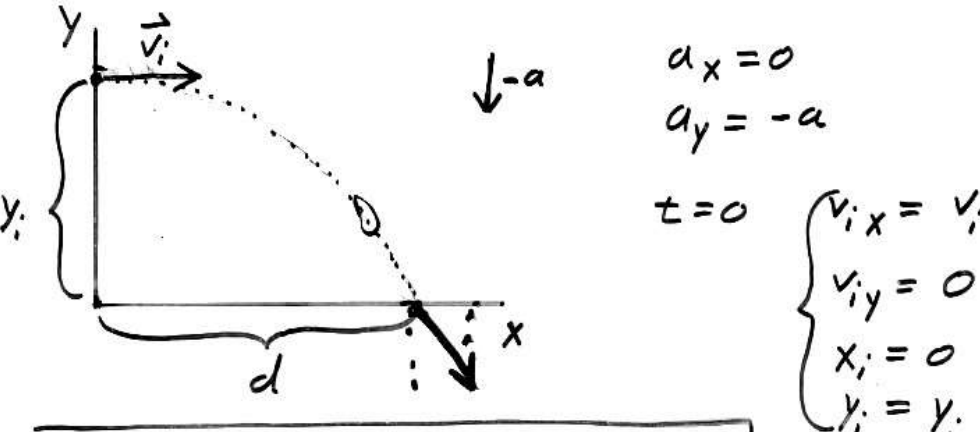
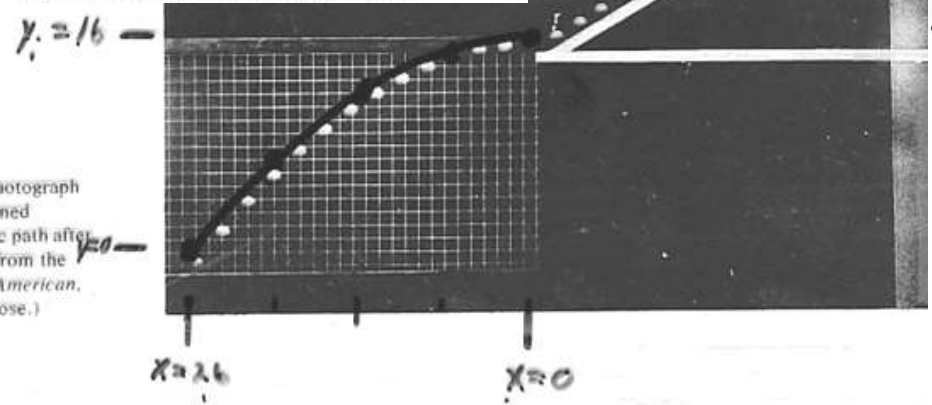
x – motion = track moves at constant velocity



A. OBVIOUSLY BACK IN CART



Figure 4-3 A multi-flash photograph of a ball rolling down an inclined plane and following a parabolic path after being projected horizontally from the foot of the plane. (Scientific American, March 1975. Photo by Ben Rose.)



$$v_x = v_i \quad x = v_i t \quad (1)$$

$$v_y = -at \quad y = y_i - \frac{a}{2} t^2 \quad (2)$$

$x$	$x_{exp}$	$y$	$y_{exp}$
0	0	$y_i = 16$	16
$d/4$	6.5	$y_i \cdot 15/16 = 15$	15
$d/2$	13	$y_i \cdot 3/4 = 12$	12
$3d/4$	19.5	$y_i \cdot 7/16 = 7$	7
$d$	26	0	0

$$y = y_i \left[ 1 - \frac{x^2}{d^2} \right]$$

• What is path in  $x, y$  plane  
 solve (1) for  $t$   $t = x/v_i$

put into (2)  $y = y_i - \frac{a}{2} \left( \frac{x}{v_i} \right)^2$

use the fact  $x=d$  when  $y=0$

$$0 = y_i - \frac{a}{2} \frac{d^2}{v_i^2}$$

$$\Rightarrow \frac{a}{2} v_i^2 = y_i/d^2$$

$$\therefore y = y_i \left[ 1 - x^2/d^2 \right]$$

3-13-extra

