# Friction : A Simple Case Study

Consider an object moving in one dimension projected with initial velocity $v_0$ (in the positive direction) at time t= 0. If no friction is present then $v = v_0$ at all times t.
If friction is present the object will slow down. Three types of frictional forces will be considered.

I] A constant frictional force ($\underline{b}$) of the sort encountered when the object is sliding over a surface. The constant, $\underline{b}$, has the units of N.

$$F = -\underline{b}$$

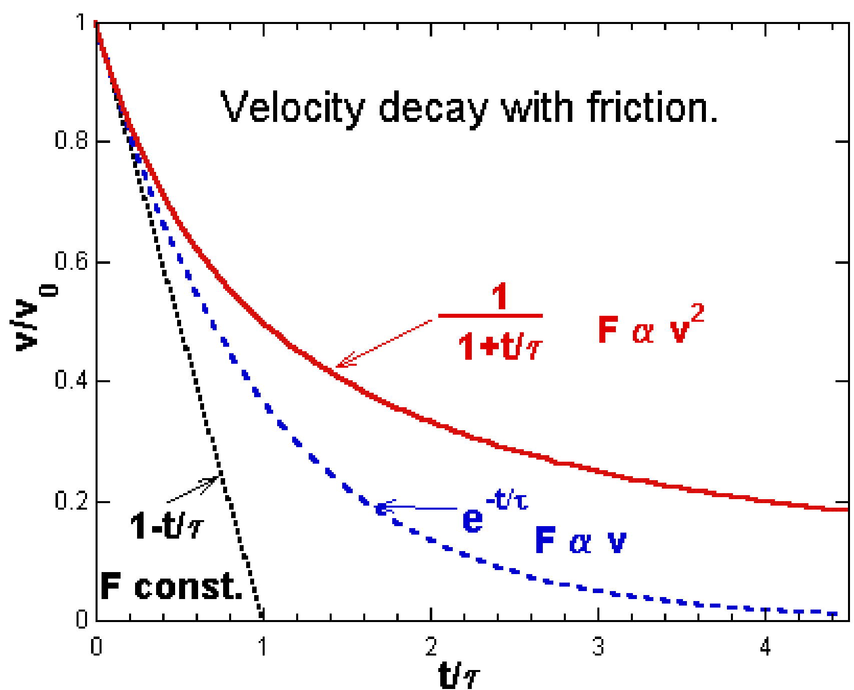
II] A frictional drag force which varies linearly with the velocity. The constant, $\underline{b}_1$, has the units of Ns/m or Kg/s.

$$F = -\underline{b}_1 v$$

III] A frictional drag force which varies quadratically with the velocity (as is often the case for higher velocity objects traveling in air). The constant, $\underline{b}_2$, has the units of Kg/m.

$$F = -\underline{b}_2 v^2$$

As can be seen from the calculations that follow (and the graph) the function al form of the velocity decay in these 3 cases is very different.




**On the next 3 pages the calculus based solutions for these cases are shown.**

**Case I**

I] A constant frictional force (b) of the sort encountered when the object is sliding over a surface. The constant, b, has the units of N-Newton's.
Reformulating the constant b has the units of acceleration ($m/s^2$).
This is the familiar case of sliding friction on a surface where the friction is equal to coefficient of kinetic friction ($\mu_k$) times the normal force (N). In the case of an object on a flat horizontal plane N = mg so:

$$\underline{b} = \mu_k N \qquad b = \mu_k g$$

The general solution of this constant frictional force problem goes as follows.

$$F = -\underline{b}$$

$$ma = -\underline{b} \qquad a = -\frac{\underline{b}}{m} = -b$$

$$\frac{dv}{dt} = -\frac{\underline{b}}{m} = -b$$

$$dv = -bdt$$

$$\int_{v_0}^{v} dv = -\int_0^t bdt$$

$$v - v_0 = -bt$$

$$v = v_0 - bt \qquad \frac{v}{v_0} = 1 - \frac{b}{v_0}t = 1 - \frac{t}{\tau}$$

$$\frac{dx}{dt} = v_0 - bt \qquad \tau = \frac{v_0}{b}$$

$$x = v_0 t - \frac{b}{2}t^2$$

# Case II

II] A frictional drag force which varies linearly with the velocity.
The constant, $\underline{b}_1$, has the units of Ns/m or Kg/s.
Using Newton's Law this can be reformulated with the constant $b_1$ has the units of 1/s.
It is sometimes useful to parameterize the friction with in a way that takes the units into account by $b_1 = 1/\tau$ .
Here $\tau$ is a decay time constant and has the units of sec.

The general solution to this problem goes as follows.

$$F = -\underline{b}_1 v$$

$$ma = -\underline{b}_1 v \qquad a = -\frac{\underline{b}_1}{m} v = -b_1 v$$

$$\frac{dv}{dt} = -\frac{\underline{b}_1}{m} v = -b_1 v = -\frac{v}{\tau} \qquad b_1 = \frac{1}{\tau}$$

$$\frac{dv}{v} = -b_1 dt$$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_{0}^{t} b_1 dt$$

$$\ln(v) - \ln(v_o) = -b_1 t$$

$$\ln\left(\frac{v}{v_o}\right) = -b_1 t$$

$$\frac{v}{v_o} = e^{-b_1 t}$$

$$\mathbf{v = \frac{dx}{dt} = v_0\, e^{t/\tau}}$$

$$\mathbf{x - x_0 = \int_0^t v_0\, e^{t/\tau}\, dt}$$

$$\mathbf{x - x_0 = -v_0 \tau\, e^{t/\tau} \Big|_0^t}$$

$$\mathbf{x - x_0 = v_0 \tau\, (1 - e^{t/\tau})}$$

# Case III

$$F = -\underline{b}_2 v^2 \qquad \underline{b}_2 = \frac{CA\rho}{2} \qquad L = \frac{2m}{CA\rho} = \frac{1}{b_2}$$

III] A frictional drag force which varies quadratically with the velocity (as is often the case for higher velocity objects traveling in air). Here the constant, $\underline{b}_2$, has the units of Kg/m.
Reformulating one has $b_2$, with the units of 1/m and the decay length L has the units of m.
For an object moving through a medium ( air) ρ is the medium density (in $Kg/.m^3$), C is the drag coefficient and A is the cross-section of the object perpendicular to the motion.

The general solution to this problem is as follows.

$$ma = -\underline{b}_2 v^2 \qquad a = -\frac{\underline{b}_2}{m} v^2 = -b_2 v^2 = -\frac{v^2}{L}$$

$$\frac{dv}{dt} = -\frac{\underline{b}_2}{m} v^2 = -b_2 v^2$$

$$\frac{dv}{v^2} = -b_2 dt$$

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\int_0^t b_2 dt$$

$$\frac{1}{v} - \frac{1}{v_o} = -b_2 t$$

$$\frac{v}{v_o} = \frac{1}{1 + v_o b_2 t}$$

$$\frac{v}{v_o} = \frac{1}{1 + \frac{v_o}{L} t} = \frac{1}{1 + \frac{t}{\tau}}$$

$$\frac{1}{v_o}\frac{dx}{dt} = \frac{1}{1 + v_o b_2 t}$$

$$\frac{dx}{dt} = \frac{v_o}{1 + v_o b_2 t}$$

$$\int_0^x dx = \int_0^t \frac{v_o}{1 + v_o b_2 t} dt$$

$$x = \frac{1}{b_2} \ln([1 + v_o b_2 t])$$

$$x b_2 = \ln([1 + v_o b_2 t])$$

$$e^{x b_2} = [1 + v_o b_2 t]$$

$$\frac{v}{v_o} = e^{-x b_2} = e^{-\frac{x}{L}}$$