Time reversal and the symplectic symmetry of the electron spin.

Rebecca Flint, M. Dzero and P. Coleman

Center for Materials Theory, Rutgers University, Piscataway, NJ 08855, U.S.A.

In the search for new kinds of collective phenomena in quantum matter, the physicist is often frustrated by the absence of natural small parameters to develop controlled approximation schemes. A theoretical tool of proven utility in this situation is the "large N" expansion, which extracts the collective physics of the real world by mapping it onto a solvable universe where particles carry a very large number (N) of spin components. This is a kind of synthetic quantum mechanics where $1/N$ plays the role of Planck's constant and quantum fluctuations about the emergent order are systematically reintroduced as a power series in this "small" parameter. In practice, the extrapolation to finite N is delicate, and an absolutely minimal requirement for success is that the large N limit respects key physical symmetries. Here we show that two vital discrete symmetries of electrons, the inversion of spin under time reversal, and the neutrality of spins under particle-hole transformations, are violated in a large class of $1/N$ expansions based on the $SU(N)$ group. Loss of these symmetries means that the method can not consistently describe singlet superconductivity or frustrated antiferromagnetism. By identifying time reversal symmetry with the symplectic symmetry of the physical electron spin, we have developed a large N expansion that respects both time-reversal and particle-hole invariance of the electron spin operator. The utility of this approach is illustrated with two model applications to frustrated antiferromagnetism and heavy fermion superconductivity. We also show how our method provides the basis for a controlled large N expansion for the resonating valence bond theory of high temperature superconductivity.

Large N approximations for interacting electron systems are carried out by generalizing the number of fermion spin components from two to N. This approach, which owes its antecedence to large N expansions for quantum chromodynamics[1], has proven invaluable in condensed matter physics, where it has helped to shed light on the physics of heavy fermion materials[2–4] and low dimensional magnetism[5, 6]. In each of these applications, the large N expansion preserves the continuous spin rotation symmetry of the original model, as an $SU(N)$ group.

One of the drawbacks of this procedure, is that two discrete symmetries of the electron spin: spin inversion under time reversal $\vec{S} \xrightarrow{\theta} -\vec{S}$, and spin invariance under particle-hole transformations, or "charge conjugation" $\mathcal{C}$, $\vec{S} \xrightarrow{\mathcal{C}} \vec{S}$ are not preserved by the spins of $SU(N)$. These symmetries occupy a place of great importance in the theory of magnetism and superconductivity. For example, the inversion of spins under time reversal is central to the formation of singlet pairs, formed through the combination of a spin with its time reversed twin. In the $SU(N)$ group, for $N>2$, the loss of this symmetry means there there is no way to form singlet pairs of particles. By contrast, the invariance of spin under "charge conjugation" defines its essential neutrality, a key property for the separation of collective charge and spin degrees of freedom. In this paper we show how these two symmetries are restored to the large N expansion by identifying the time-reversal symmetry of spins with an underlying symplectic symmetry.

We begin by exposing the link between the time-inversion of spins and symplectic symmetry. Time reversal is an antiunitary operator θ that acts on an electron wavefunction $\psi_\sigma(x,t)$ as $\theta\psi(\mathbf{x},t) = i\sigma_2\psi^*(\mathbf{x},-t)$ where σ_2 is the second Pauli matrix and $\psi^* \equiv (\psi^\dagger)^T$ is the complex conjugate of the wavefunction. θ is normally written as the product $\theta = i\sigma_2 K$ [7] of the skew-symmetric matrix $i\sigma_2$ and the complex conjugation operator K. K conjugates the amplitudes of a quantum state, so $Ka|\psi\rangle = a^* K|\psi\rangle$. The requirement that time reversal is rotationally invariant, $U\theta = \theta U$, implies

$$U\theta U^\dagger = \theta, \tag{1}$$

where U is a unitary rotation operator. Writing $\theta = i\sigma_2 K$, and noting that the charge conjugation operator converts $U^\dagger$ to the transpose U^T, $KU^\dagger = U^T K$, it follows that

$$U i\sigma_2 U^T = i\sigma_2.$$

This is a "symplectic" transformation, a class of transformation that preserves an antisymmetric matrix, in this case $i\sigma_2$. The mysterious appearance of the transpose U^T, rather than the usual Hermitian conjugate $U^\dagger$ can be understood as a consequence of the anti-unitary nature of time reversal. By expanding the symplectic condition for infinitesimal rotations, $U = 1 + i\vec{\alpha}\cdot\vec{S}$, we obtain the inversion of spins under time reversal $\vec{S} \xrightarrow{\theta} \theta\vec{S}\theta^{-1} = \sigma_2\vec{S}^T\sigma_2 = -\vec{S}$.

To extend time reversal symmetry to the large N expansion, we need to consider fermionic or boson spinors with an even number $N = 2k$ of spin components, $(\widetilde{\psi})_\alpha = \left(\psi_1 \ \psi_{-1} \ldots \ \psi_k \ \psi_{-k}\right)$. The natural N-component generalization of the time reversal operation is given by $\theta = i\sigma_2 K$ where σ_2 is now the N dimensional skew-symmetric matrix $[i\sigma_2]_{\alpha\beta} = \tilde{\alpha}\delta_{\alpha,-\beta}$, where $\tilde{\alpha} = sgn(\alpha)$. The time reversal condition (1) then becomes $Ui\sigma_2 U^T = i\sigma_2$. This condition defines the large subgroup of the $SU(N)$ group called $SP(N)$ whose symplectic spins invert under time reversal $\sigma_2 S^T \sigma_2 = -S$. Fifteen years ago, Read and Sachdev[8] made the pioneering observation that the symplectic group $SP(N)$ allows spin operators that form singlet pairs. This approach has been extensively applied to frustrated magnetism[9, 10] and more recently, to paired Fermi gases[8, 11, 12]. From our discussion, the symplectic group assumes a new importance, not merely as a way to form two particle singlets, but as a unique way to sustain a consistent definition of time-inversion symmetry in the large N limit. As we will see, this more general requirement makes a distinction between Hamiltonians with $SP(N)$ symmetry, and a more restrictive class of Hamiltonians built exclusively from symplectic spins.

The spin operators of the $SU(N)$ group are written

$$\mathcal{S}_{\alpha\beta} = \psi^\dagger_\alpha \psi_\beta - \left(\frac{n_\psi}{N}\right)\delta_{\alpha\beta}, \tag{2}$$

where $n_\psi = \sum_\alpha \psi^\dagger_\alpha \psi_\alpha$ is the number of particles that make up the spin. Under time reversal, $\mathcal{S}_{\alpha\beta} \xrightarrow{\theta} \tilde{\alpha}\tilde{\beta}\mathcal{S}_{-\beta,-\alpha}$, $SU(N)$ spins have no well defined parity. When ψ_α is a Fermi operator, we can also define a charge conjugation operator $\mathcal{C}$ that converts particles into holes $\psi_\alpha \xrightarrow{\mathcal{C}} \tilde{\alpha}\psi^\dagger_{-\alpha}$. The $SU(N)$ spin operator inverts under the combined operation $\mathcal{C}\theta$, $\mathcal{S}_{\alpha\beta} \xrightarrow{\mathcal{C}\theta} -\mathcal{S}_{\alpha\beta}$ so that when the time reversal parity is ill-defined, the neutrality of the spin is also ill-defined.

By taking antisymmetric or symmetric combinations of the $SU(N)$ spins with their time-reversed version, the $SU(N)$ spins divide into two groups

$$\text{“magnetic” moments} \qquad S_{\alpha\beta} = \psi^\dagger_\alpha \psi_\beta - \tilde{\alpha}\tilde{\beta}\psi^\dagger_{-\beta}\psi_{-\alpha}, \qquad (\theta, \mathcal{C}) = (-, +) \tag{3}$$

which invert under time reversal and are invariant under charge conjugation, together with

$$\text{“electric” dipoles} \qquad \mathcal{P}_{\alpha\beta} = \psi^\dagger_\alpha \psi_\beta + \tilde{\alpha}\tilde{\beta}\psi^\dagger_{-\beta}\psi_{-\alpha}, \qquad (\theta, \mathcal{C}) = (+, -) \tag{4}$$

so named, because they are invariant under time reversal, but reverse under charge conjugation[13]. There are $D_N = \frac{1}{2}N(N+1)$ magnetic moments that form the generators of $SP(N)$ and $d_N = \frac{1}{2}(N-2)(N+1)$ electric dipoles that make up the rest of the group. For the physical case of $N = 2$, there are no dipoles, but as N becomes large, the $SU(N)$ group contains approximately equal numbers of moments and unphysical dipoles.

The magnetic components of the $SU(N)$ spins, can be visualized as lying in a D_N dimensional "symplectic plane" as shown in Fig. 1. Since they form a closed $SP(N)$ algebra, a Hamiltonian $H[S]$ composed exclusively of magnetic spin operators will develop spin dynamics $\frac{dS}{dt} = -i[H[S], S]$ that evolve exclusively within the symplectic plane. This requirement is more stringent than the requirement that the Hamiltonian display $SP(N)$ symmetry. However, if the Hamiltonian $H[S, \mathcal{P}]$ includes any "dipole moment" operators $\mathcal{P}$, the non-trivial commutators between the magnetic moments and electric dipoles destroy the closure and time reversal ceases to exist as a well-defined parity. It is thus highly desirable to take the large N limit using Hamiltonians that completely eliminate any dipole component of the interactions. We call the new strategy of using exclusively magnetic spins in the interaction, "symplectic N." We now discuss two applications of this approach, to frustrated magnetism and heavy electron superconductivity.

Applications to quantum magnetism In quantum magnetism, one generally uses a Schwinger boson spin representation, so that $S_{\alpha\beta} = b^\dagger_\alpha b_\beta - \tilde{\alpha}\tilde{\beta} b^\dagger_{-\beta} b_\alpha$ where b is an N component Schwinger boson[6]. For convenience, we shall denote the set of magnetic spin operators by the vector $\vec{S} = (\hat{S}^1, \ldots \hat{S}^D)$, with the understanding that a dot product means the combination $S(i) \cdot S(j) \equiv \frac{1}{2} \sum_{\alpha,\beta} S_{\alpha\beta}(i) S_{\beta\alpha}(j)$. The spin Casimir is $(\vec{S}_j)^2 = n_j(n_j + N)$, where n_j is the number of bosons at site j, thus the symplectic spin representation is fixed by $n_j = NS$. Using the explicit form of the symplectic spins, we find the Heisenberg interaction decouples into two terms

$$J\vec{S}_i \cdot \vec{S}_j = J[A^\dagger_{21} A_{21} - B^\dagger_{21} B_{21}], \tag{5}$$

where $B^\dagger_{21} = \sum_\sigma \tilde{\sigma} b^\dagger_{2\sigma} b^\dagger_{1-\sigma}$ creates a valence bond between sites while $A_{21} = \sum_\sigma b^\dagger_{2\sigma} b_{1\sigma}$ "resonates" valence bonds between sites. Symplectic invariance requires that we treat these two terms in equal measure. Earlier $SP(N)$ approaches to frustrated magnetism[8–10] treated interactions of the form $-JB^\dagger_{12} B_{12}$. While this interaction is an $SP(N)$ singlet, it contains a mix of magnetic moments and electric dipoles $-2B^\dagger_{12} B_{12} = \vec{S}_1 \cdot \vec{S}_2 - \vec{P}_1 \cdot \vec{P}_2$ that spoils the closure of the magnetic dynamics.

When we carry out a Hubbard Stratonovich factorization of the Heisenberg interaction (5), it separates into two amplitudes h and Δ describing bond resonance and condensation respectively

$$J\vec{S}_1 \cdot \vec{S}_2 = \left(b^\dagger_{2\sigma}, \tilde{\sigma} b_{2\ -\sigma}\right) \begin{bmatrix} h & \Delta \\ \bar{\Delta} & \bar{h} \end{bmatrix} \begin{pmatrix} b_{1\sigma} \\ \tilde{\sigma} b^\dagger_{1\ -\sigma} \end{pmatrix} + \frac{N}{J}(|\Delta|^2 - |h|^2) \tag{6}$$

This kind of decoupling scheme was first proposed by Ceccato, Gazza and Trumper[14, 15] for $SU(2)$ spin systems, where it is one of many alternative decoupling procedures. In symplectic N it is the unique form which preserves the time reversal parity of the spins.

A classic test application for new approaches in frustrated magnetism is the $J_1 - J_2$ Heisenberg model on a square lattice [16]

$$H = J_1 \sum_{\mathbf{x},\mu} \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}+\mu} + J_2 \sum_{\mathbf{x},\mu'} \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}+\mu'}, \tag{7}$$

where J_1 and J_2 are the first and next nearest neigbor couplings. A variant of this basic model is of current interest in the context of field-tuned dimer spin systems. At large but finite J_2, this model describes two interpenetrating Néel states that are classically decoupled. When fluctuations are included, a biquadratic interaction locks the two sublattices together in a collinear configuration. In early work, Chandra, Coleman and Larkin[17] showed that this effect gives rise to a finite temperature Ising transition associated with the development of the singlet bond variable

$$\sigma = \frac{1}{2S^2}\left(\vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}+\mathbf{e}_x} - \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}+\mathbf{e}_y}\right) \tag{8}$$

Capriotti et al.[18] found that the Ising transition temperature rises from zero at $J_1 = 0$ and vanishes shortly before the transition from two sublattice to Néel order at $J_1 = 2J_2$. Despite the singlet character of the Ising operator, previous attempts to model order-from-disorder using an RVB description have been unable to reproduce this behavior. The symplectic N scheme recovers the finite temperature Ising transition as a resonating valence bond instability. When J_1/J_2 is small, a boson pairing field develops across the diagonal bonds at high temperatures $T \sim J_2$, describing the development of antiferromagnetic correlations within each of the sublattices. (Fig. 2). At still lower temperatures, a valence bond instability with two unstable eigenmodes develops, given by $\phi_{RVB1} = (-h_x, \Delta_y)\,, \phi_{RVB2} = (-h_y, \Delta_x)$ These two unstable eigenmodes can be identified with the Ising order

$$\sigma \propto \phi^2_{RVB1} - \phi^2_{RVB2} = (h_x^2 + \Delta_y^2) - (h_y^2 + \Delta_x^2), \tag{9}$$

and the corresponding resonating valence bond instability is identified as the Ising phase transition

$$T_I \equiv T_{RVB} = \frac{4\pi J_2 S^2}{\log\left(\frac{2J_2 S}{J_1\sqrt{2\gamma}}\right)} \tag{10}$$

Apart from a numerical factor of S inside the logarithm, this is precisely the result obtained by Chandra, Coleman and Larkin for $J_1 \ll J_2$. The full behavior of T_I can be calculated with symplectic N (Fig. 2) and qualitatively reproduces that of Capriotti et al[28]. The existence of a well-defined spin time reversal parity appears to play an essential role in the physics, and the introduction of even a small component of antisymplectic spins into the Heisenberg interaction falsely predicts a first order transition as $J_1/J_2 \to 2$.

Symplectic symmetry and superconductivity.

For metals and superconductors, it is more convenient to represent spins using an pseudo-fermion representation, $\hat{S} = f^\dagger_\alpha \vec{S}_{\alpha\beta} f_\beta$. As discussed above, for fermions, the restoration of a well-defined spin time reversal parity also restores the neutrality of the spin operators under the charge conjugation transformation $f_\sigma \to -\tilde{\sigma} f^\dagger_{-\sigma}$. In fact, charge conjugation symmetry is actually manifested as a continuous $SU(2)$ gauge symmetry

$$f_\sigma \to \cos\theta e^{i\phi} f_\sigma + \tilde{\sigma} \sin\theta e^{-i\phi} f^\dagger_{-\sigma}. \tag{11}$$

This gauge symmetry was discovered by Affleck et al[19] for spin 1/2 operators: remarkably, it survives as a symmetry of *all* symplectic spin operators. As a mathematical tool, the $SU(2)$ gauge symmetry plays an essential role in the RVB description of strongly correlated superconductors, allowing a description of the development of valence bonds in a spin fluid, and the transmission of their pair correlations to the electron sea[20, 21]. For the first time, this physics can now be explored within a controlled large N approach.

As an example of this physics in action, we now discuss the application of symplectic N to heavy fermion superconductivity. To date, there are no controlled model calculations that unify the Kondo lattice physics with the development of superconductivity.

The recent discovery of superconductivity below $18.5K$ in $PuCoGa_5$[22] provides an extreme example of this phenomenon, for in this heavy fermion material superconductivity and spin quenching take place simultaneously, without the prior establishment of a paramagnetic heavy fermi liquid. Ten years ago, Coleman, Andrei, Tsvelik and Kee (CATK)[23] proposed that heavy electron superconductivity might develop as part of the Kondo effect, in cases where there is a constructive interference between two competing spin-screening channels, leading to the formation of a composite order parameter Λ_j involving the bound-state of a triplet pair and a spin-flip

$$\Lambda_j = \langle \Psi_{N-2} | \psi^1_\downarrow(j) \psi^2_\downarrow(j) S^+_f(j) | \Psi_N \rangle. \qquad (12)$$

where $\psi^\Gamma_\alpha(j) = \frac{1}{\mathcal{N}} \sum_{\vec{k}} \gamma^\Gamma_{\mathbf{k}} c_{\mathbf{k}\alpha} e^{-i\mathbf{k}\cdot\mathbf{x}_j}$ annihilates an electron in the Wannier state with symmetry Γ at a given site j, $\mathcal{N}$ is the number sites in the lattice and $S^+_f(x_j) \equiv S^+(j)$ raises the spin of the localized f-moment at site $\mathbf{x}_j$. Such a model provides an interesting candidate for $PuCoGa_5$, however, demonstration of the CATK mechanism relied heavily on the $SU(2)$ gauge symmetry of localized moments, a feature that could not be treated in a controlled calculation. The validity of the mechanism and its application to real systems has since remained in doubt. Using symplectic N, the CATK mechanism can now be demonstrated in a controlled large N limit.

The CATK model takes the form of a two-channel Kondo lattice

$$H = \sum \epsilon_{\mathbf{k}\sigma} c^\dagger{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} J_{\mathbf{k},\mathbf{k}'} c^\dagger{}_{\mathbf{k}\alpha} \vec{S}_{\alpha\beta} c_{\mathbf{k}'\beta} \cdot \mathbf{S}_j \; e^{(\mathbf{k}-\mathbf{k}')\cdot\vec{R}_j} + \frac{1}{N} \sum_{j,\Gamma} J_\Gamma \vec{S}_\Gamma(j) \cdot \vec{S}_f(j), \qquad (13)$$

where $\vec{S}_\Gamma(j) = \psi^{\Gamma\dagger}(\mathbf{x}_j) \vec{S} \psi^\Gamma(\mathbf{x}_j)$ $(\Gamma = 1, 2)$ is the component of the conduction electron spin density with symmetry Γ at site j. The operator $\vec{S}_f(j) = f^\dagger_j \vec{S} f_j$ is the f-electron spin at site j. In the envisioned application to $PuCoGa_5$, these two screening channels would derive from virtual valence fluctuations of the magnetic Pu ions into the non-magnetic f^6 and the f^4 configurations, respectively $f^4 + e^-_{\Gamma=2} \rightleftharpoons f^5 \rightleftharpoons f^6 + h^+_{\Gamma=1}$ where $h^+_{\Gamma=1}$ and $e^-_{\Gamma=2}$ represent electrons and holes in the two symmetry channels. It is the crystal field symmetries of the f^4 and f^5 states which determine the symmetries of the two competing screening channels [24].

We expand the Kondo interaction $H_\Gamma(j) = \frac{J_\Gamma}{N} \vec{S}_\Gamma(j) \cdot \vec{S}_f(j)$ in terms of symplectic spin operators using (3). When this interaction is factorized, it decouples into a Kondo hybridization V and pairing field Δ as follows[23, 25]

$$H_\Gamma(j) \to \sum_\sigma \left[\left(f^\dagger_\sigma V^\Gamma + \tilde{\sigma} f_{-\sigma} \Delta^\Gamma \right) \psi_{\Gamma\sigma} + \text{H.C} \right] + N \left(\frac{|V^\Gamma|^2 + |\Delta^\Gamma|^2}{J_\Gamma} \right) \qquad (14)$$

At each site j in the lattice, it is convenient to define a Nambu spinor for the f-electrons and two corresponding matrix $SU(2)$ order parameters

$$\tilde{f}_{j\sigma} = \begin{pmatrix} f_{j\sigma} \\ \tilde{\sigma} f^{\dagger}_{j-\sigma} \end{pmatrix}, \qquad \mathcal{V}_{\Gamma j} = \begin{pmatrix} V^{\Gamma} & \bar{\Delta}^{\Gamma} \\ \Delta^{\Gamma} & -\bar{V}^{\Gamma} \end{pmatrix}_j . \tag{15}$$

The Hamiltonian is invariant under the local gauge transformations $\tilde{f}_{j\sigma} \to g_j \tilde{f}_{j\sigma}$, $\mathcal{V}_{\Gamma j} \to g_j \mathcal{V}_{\Gamma j}$ where a single $SU(2)$ matrix g_j transforms the order parameter in both channels at each site.

The formation of Kondo singlets in a single channel does not lead to superconductivity, since an $SU(2)$ gauge transformation on the f-electron can always absorb the pairing term Δ into a redefinition of the f-electron: $(f^{\dagger}_{\sigma} V^{\Gamma} + \tilde{\sigma} f_{-\sigma} \Delta^{\Gamma}) \to V_0 \tilde{f}^{\dagger}_{\sigma}$. However, the product $\Psi = \mathcal{V}^{\dagger}_{2j} \mathcal{V}_{1j}$ is an $SU(2)$ invariant. The off-diagonal components of Ψ are directly proportional to the composite order parameter (15) $\Lambda_j \propto \Psi_{21}(j) = (V_{1j}\Delta_{2j} - V_{2j}\Delta_{1j})$. In a single impurity model, this order parameter is forbidden, because it breaks channel symmetry, but in the lattice, electrons travelling between sites no longer conserve the channel index, and this permits composite order to develop.

We have computed the superconducting transition temperature T_c as a function of the ratio J_2/J_1 using symplectic N[28]. Fig. 3 (a) shows the results of a model calculation for a two-dimensional Kondo lattice, assuming uniform expectation values for $\mathcal{V}_{\Gamma}$. $J_1 = J_2$. It is instructive to contrast the phase diagrams of the $SU(N)$ and symplectic large N limits. In the former, there is a single quantum phase transition that separates the heavy electron Fermi liquids formed via a Kondo effect about the strongest channel. In the symplectic treatment, this quantum critical point is immersed beneath a superconducting dome. This is, to our knowledge, the first *controlled* mean-field theory in which the phenemenon of "avoided criticality" gives rise to superconductivity.

We end with a brief discussion of the application of our approach to the problem of high temperature superconductivity. A central arena for discussion of high temperature superconductivity concerns the t-J model,

$$H = -t \sum_{(ij),\sigma} (X_{\sigma 0}(i) X_{0\sigma}(j) + \text{H.c}) + J \sum_{(ij)} \vec{S}_i \cdot \vec{S}_j$$

where $X_{0\sigma}(j)$ is the Hubbard operator that creates a hole at site j, while imposing the constraint of no double occupancy. This model has played a central role in discussions of the RVB theory of superconductivity[20, 21], which depends essentially on the invariance of the spin operators under charge conjugation, features inaccessible to a conventional large N expansion.

The symplectic generalization of Hubbard operators is given by

$$X_{\sigma 0} = f^\dagger_\sigma \hat{b}_1 + \tilde{\sigma} f_{-\sigma} \hat{b}_2 \tag{16}$$

where $\hat{b}_1$ and $\hat{b}_2$ are respectively, positively and negatively charged slave boson fields. The anticommutator of these operators generates a symplectic, rather than an $SU(N)$ spin

$$\{X_{\sigma 0}, X_{0\sigma'}\} = \delta_{\sigma\sigma'} X_{00} + S_{\sigma\sigma'} \tag{17}$$

where $X_{00} = n_a + n_b + 1$ and $S_{\sigma\sigma'}$ is the symplectic spin operator, which guarantees that the charge dynamics of this model respect the symplectic symmetry. This representation of the t-J model was first formulated by Wen and Lee[26], as a way of preserving local $SU(2)$ charge conjugation symmetry. By identifying the symplectic symmetry of the Wen-Lee formulation of the Hubbard operators, one is now in a position to provide the first large N expansion for the RVB theory of superconductivity.

This research was supported by the National Science Foundation grant DMR-0605935 and DE-FE02-00ER45790 (M. Dzero). The authors would like to thank N. Andrei, K. Basu, E. Miranda, P. Pagliuso, S. Sachdev, N. Read, G. Zarand and particularly Scott Thomas, for discussions related to this work.

[1] E. Witten, Nucl Phys. **B** 160, 57 (1979).

[2] N. Read, and D. M. Newns, *J. Phys. C* **29**, L1055, (1983).

[3] N. Read and D.M. Newns, *J. Phys. C* **16**, 3273 (1983).

[4] A. Auerbach, and K. Levin, *Phys. Rev. Lett.* **57**, 877 (1986).

[5] I. Affleck and J. B. Marston, *Phys. Rev. B*, 37, 3774–3777, (1988).

[6] D. P. Arovas and A. Auerbach, *Phys. Rev. B*, **38**, 316, (1988).

[7] J. J. Sakurai, Modern Quantum Mechanics Revised Edition, pp 277, (Addison Wesley), (1994).

[8] N. Read and Subir Sachdev, *Phys. Rev. Lett.* **66**, 1773 (1991); Subir Sachdev and Ziquiang Wang, *Phys. Rev.* B **43**, 10229 (1991).

[9] O. Tchenyshyov, R. Moessner and S. L. Sondhi, *Europhysics Letters* 73, 278-284 (2006).

[10] S. Sachdev, *Phys. Rev.* **45**, 12377 (1992).

[11] P. Nickolić and S. Sachdev, *Phys. Rev.* A **75**, 033608 (2007).

[12] M. Veillette, D. Sheehy and L. Radzihovsky, *Phys. Rev.* A **75**, 043614 (2007).

[13] C. Wu and S. Zhang, Phys. Rev. B **71**, 155115 (2005).

[14] H.A. Ceccatto, C.J. Gazza and A.E. Trumper, *Phys. Rev. B* **47**, 12329 - 12332 (1993).

[15] A. E. Trumper, L. O. Manuel, C. J. Gazza, and H. A. Ceccatto, *Phys. Rev. Lett.* **78**, 2216 (1997).

[16] P. Chandra and B. Doucot, *Phys. Rev.* B**38**, 9335, 1988

[17] P. Chandra, P. Coleman and A. I. Larkin, *Phys. Rev. Lett.* **64**, 88 (1990).

[18] C. Weber, L. Capriotti, G. Misguich, F. Becca, M. Elhajal and F. Mila, *Phys. Rev. Lett.* **91**, 177202 (2003). ; L. Capriotti, T. Roscilde, A. Fubini and V. Tognetti, *Phys. Rev. Lett.* **92**, 157202 (2004).

[19] I. Affleck, Z. Zou, T. Hsu and P. W. Anderson, *Phys. Rev. B* **38**, 745, (1988).

[20] P. W. Anderson, G. Baskaran, Z. Zou and T. Hsu, *Phys. Rev. Lett.* **58**, 2790 (1987),

[21] G. Kotliar and J. Liu, *Phys. Rev. B* **38**, 5142 (1988).

[22] J. L. Sarrao *et al.*, Nature (London) **420**, 297 (2002).

[23] P. Coleman, A. M. Tsvelik, N. Andrei and H. Y. Kee, *Phys. Rev. B* **60**, 3608 (1999).

[24] D. L. Cox and M. Jarrel, J. Phys.: Condens. Matter **8**, 9825 (1996).

[25] Y. Ran and X.G. Wen, cond-mat/0609620v3 (2006).

[26] X.G. Wen and P. Lee, *Phys. Rev. Lett.* **76**, 503 (1996).

[27] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).

[28] "Supplementary Information is linked to the online version of the paper at www.nature.com/nature."

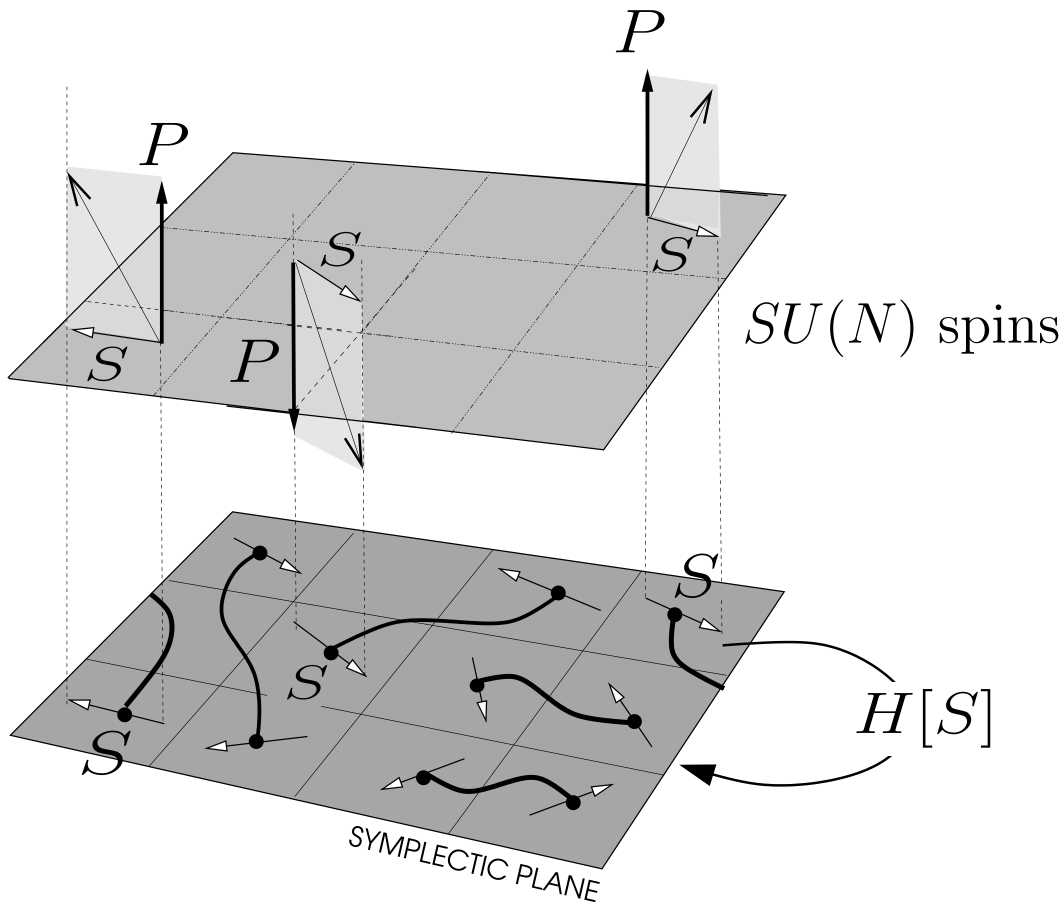

FIG. 1: Illustrating the separation of $SU(N)$ spins into magnetic and electric dipole components. Spins lying in magnetic, symplectic plane are neutral, time reverse and are able to form two-particle singlets. Dipoles pointing out of the plane do not time reverse, are not invariant under charge conjugation, and they can not form two-particle singlets. Physical properties require that the spin dynamics remains closed in the magnetic plane.

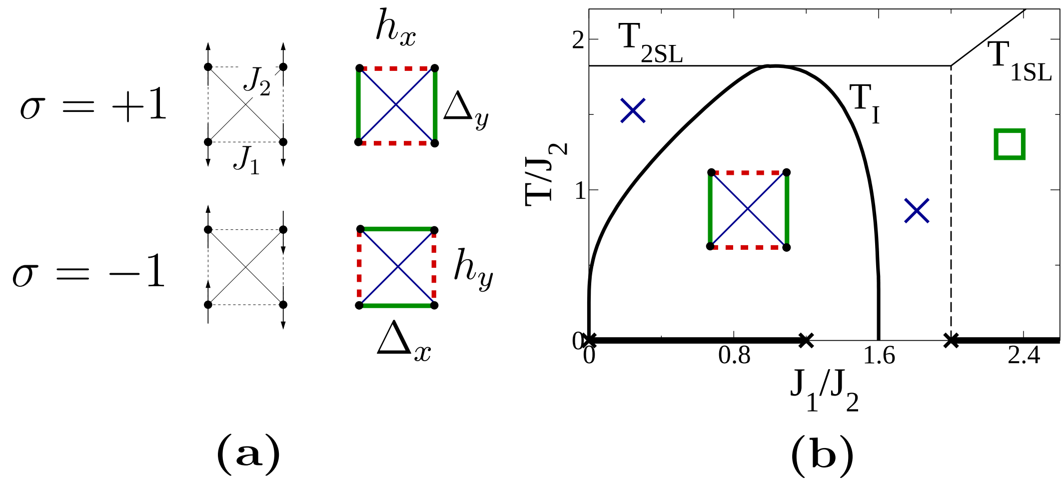

FIG. 2: (a)For small J_1/J_2, the $J_1 - J_2$ Heisenberg model develops collinear Ising order ($\sigma = \pm 1$); the equivalent valence bond configurations, ϕ_{RVB1} and ϕ_{RVB2} are shown to the right(valence bonds are indicated by a solid line, while dashed lines show bond resonance). (b) Phase diagram for $S = 1/2$, computed in the symplectic N limit. The computed second-order Ising transition temperature T_I, forms a "dome" that extends up to $J_1/J_2 \simeq 1.6$.

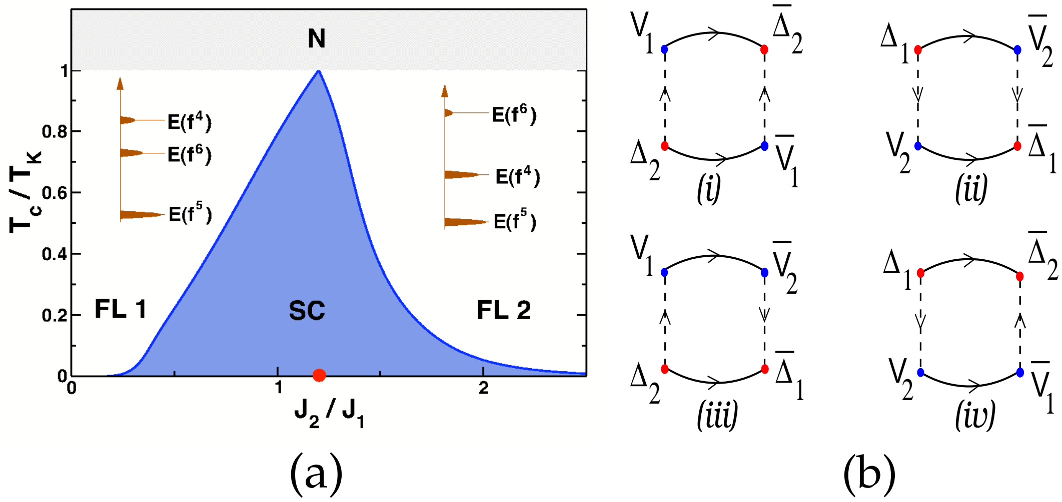

FIG. 3: (a) Phase diagram for a simple $2D$ two channel Kondo lattice, computed in the symplectic large N limit. For the $PuCoGa_5$ superconductor difference between relative configurational energies $E(f^6) - E(f^5)$ and $E(f^4) - E(f^5)$ results in either stronger (left inset) or weaker (right inset) first channel. (b) Leading Feynman diagrams responsible for composite pairing in the two-channel Kondo lattice. The sum of all four diagrams (i)- (iv) gives a contribution proportional to $|\Lambda|^2 = |(V_1\Delta_2 - V_2\Delta_1)|^2$.