- IDEA: Even though Fermions do not condense, by introducing a new kind of anticommuting number, we can define a basis of fermionic coherent states and use this to develop fermionic path integrals.

$$\left.\begin{aligned} |c\rangle &= \exp\left[\hat{c}^{\dagger} c\right] |\phi\rangle \\ \langle \bar{c}| &= \langle \phi| \exp\left[\bar{c}\, \hat{c}\right] \end{aligned}\right\}$$

If a fermionic coherent state is an eigenvalue of the destruction operator, then since $(\hat{c})^2 = 0$, it follows that $c^2 = 0$.

More generally, since fermion operators anticommute, $\{\hat{c}_x, \hat{c}_{x'}\} = 0$ we require that $c_x c_{x'} = -c_{x'} c_x$, i.e. that Grassmann numbers anticommute with each other.

$\hat{c}^{\dagger}_x |\{c\}\rangle = c_x |\{c\}\rangle$, then we require for consistency that

$$\left.\begin{aligned} \hat{c}_1 \hat{c}_2 |c\rangle &= \hat{c}_1 |c\rangle c_2 = |c\rangle c_1 c_2 \\ &= -\hat{c}_2 \hat{c}_1 |c\rangle = -|c\rangle c_2 c_1 \end{aligned}\right\} \Rightarrow c_1 c_2 = -c_2 c_1$$

$c_1 c_2 = -c_2 c_1$ $\qquad\qquad$ $\{c_x, \hat{c}_{x'}\} = \{c_x, \hat{c}^{\dagger}_{x'}\} = 0.$

$c_1^2 = c_2^2 = 0$

Grassman Calculus: Summary

- **Algebra**

$$c_1 c_2 = -c_2 c_1 \qquad c\hat{b} = \hat{b}c \qquad c\hat{\psi} = -\hat{\psi}c$$

- **Functions**

$$f(\bar{c},c) = f_0 + \bar{c} f_1 + \bar{f}_1 c + f_{12}\bar{c}c$$

- **Calculus.**

$$\partial_{\bar{c}} f = f_0 + f_1 + f_{12} c$$

$$\partial_{\bar{c}} \equiv \int d\bar{c} \qquad \cdot \int d\bar{c} = 0 ; \int d\bar{c}\,\bar{c} = 1$$

- **Coherent state**

$$|c\rangle = |\phi\rangle + \hat{c}^{+} c|\phi\rangle = |0\rangle + |1\rangle c$$

$$\langle n|c\rangle = c^{n} \qquad n = 0,1$$

- **Overcomplete**

$$\langle \bar{c}|c\rangle = e^{\bar{c}c}$$

- **Completeness**

$$\sum_{\bar{c},c} |\bar{c}\rangle\langle c| = 1 = \int d\bar{c}\,dc\, e^{-\bar{c}c}$$

- **Matrix elements**

$$:A[\hat{c}^{+}, c]: \rightarrow e^{\bar{c}c} A[\bar{c},c]$$

- **Trace formula.**

$$\mathrm{Tr}\,\hat{A} = \int d\bar{c}\,dc\, e^{-\bar{c}c} \langle -\bar{c}|A|c\rangle$$

- **Change of variable**

$$J\begin{pmatrix} c_1 \ldots c_r \\ \xi_1 \ldots \xi_r \end{pmatrix} = \left| \frac{\partial(c_1 \ldots c_r)}{\partial(\xi_1 \ldots \xi_r)} \right|^{-1}$$

$$\partial_{c_i} = \frac{\partial \xi_j}{\partial c_i} \partial_{\xi_j} \qquad \prod dc = {}_{j}\prod \partial_{c_1} \ldots \partial_{c_n}$$

$$= \prod_{i=1} \frac{\partial \xi_c}{\partial c_i} \partial_{\xi_c} = \left\{ (-1)^{P} \frac{\partial_{P_i}}{\partial c_i} \right\} \prod d\xi_i$$

$$= \left| \frac{\partial \xi}{\partial c} \right| \prod d\xi .$$

- **Gaussian Int**

$$\int \prod d\bar{c}\,dc\, e^{-[\bar{c}Ac - \bar{j}\cdot c - \bar{c}\cdot j]} = \det A\, e^{\bar{j}A^{-1}j}$$

Table 12.2 Grassman calculus.

Algebra	$c_1 c_2 = -c_2 c_1$	Anticommute with fermions and other Grassman numbers
	$c\hat{b} = \hat{b}c, \quad c\hat{\psi} = -\hat{\psi}c$	Commute with bosons, anticommute with fermi operators
Functions	$f[\bar{c}, c] = f_o + \bar{c} f_1 + \tilde{f}_1 c + f_{12}\bar{c}c$	Since $c^2 = 0$, truncate at linear order in each variable
Calculus	$\partial f = -\tilde{f}_1 - f_{12}\bar{c}$	Differentiation
	$\bar{\partial} f = f_1 + f_{12} c$	
	$\int dc \equiv \partial_c$	$\int dc 1 = \partial_c 1 = 0$ $\int dc c = \partial_c c = 1$
Completeness	$\langle c\|c\rangle = e^{\bar{c}c}$	Overcomplete basis
	$\int d\bar{c}dc e^{-\bar{c}c} \|c\rangle\langle\bar{c}\| = \underline{1}$	Completeness relation
	$\mathrm{Tr}[\hat{A}] = \int d\bar{c}dc e^{-\bar{c}c}\langle -\bar{c}\|\hat{A}\|c\rangle$	Trace formula
Change of variable	$J\left(\dfrac{c_1 \ldots c_r}{\xi_1 \ldots \xi_r}\right) = \left\|\dfrac{\partial(c_1, \ldots c_r)}{\partial(\xi_1, \ldots \xi_r)}\right\|^{-1}$	Jacobian (inverse of bosonic Jacobian)
Gaussian integrals	$\int \prod_j d\bar{c}_j dc_j e^{-\left[\bar{c}\cdot A\cdot\bar{c} - \bar{j}\cdot c - \bar{c}\cdot j\right]} = \mathrm{detA} \times e^{\left[\bar{j}\cdot A^{-1}\cdot j\right]}$	

Notice the formal parallel with the overlap of bosonic coherent states. To derive the completeness relation, we start with the identity

$$\int d\bar{c}dc e^{-\bar{c}c} c^n \bar{c}^m = \delta_{nm} \qquad (n, m = 0, 1). \tag{12.104}$$

Then, by writing $c^n = \langle n|c\rangle$, $\bar{c}^m = \langle\bar{c}|m\rangle$, we see that the overlap between the eigenstates $|n\rangle$ of definite particle number is given by

$$\delta_{nm} = \langle n|m\rangle = \int d\bar{c}dc e^{-\bar{c}c}\langle n|c\rangle\langle\bar{c}|m\rangle = \langle n| \int d\bar{c}dc e^{-\bar{c}c} \; |c\rangle\langle\bar{c}| \; |m\rangle, \tag{12.105}$$

from which it follows that

$$\int d\bar{c}dc |c\rangle\langle\bar{c}| e^{-\bar{c}c} = |0\rangle\langle 0| + |1\rangle\langle 1| \equiv \underline{1}. \tag{12.106}$$

completeness relation

Alternatively, we may write

$$\sum_{\bar{c},c} |c\rangle\langle\bar{c}| = \underline{1},$$

where

$$\sum_{\bar{c},\, c} \equiv \int d\bar{c}dce^{-\bar{c}c} \tag{12.107}$$

is the measure for fermionic coherent states. The exponential factor $e^{-\bar{c}c} = 1/\langle\bar{c}|c\rangle$ provides the normalizing factor to take account of the overcompleteness.

Matrix elements between coherent states are easy to evaluate. If an operator $A[\hat{c}^\dagger, \hat{c}]$ is normal ordered, then, since the coherent states are eigenvectors of the quantum fields, it follows that

$$\langle\bar{c}|\hat{A}|c\rangle = \langle\bar{c}|c\rangle A[\bar{c}, c] = e^{\bar{c}c}A[\bar{c}, c]. \tag{12.108}$$

That is,

$$\langle\bar{c}|\hat{A}|c\rangle = e^{\bar{c}c} \times \text{ c-number formed by replacing } A[\hat{c}^\dagger, \hat{c}] \to A[\bar{c}, c]. \tag{12.109}$$

This wonderful feature of coherent states enables us, at a swoop, to convert normal-ordered operators into c-numbers.

The last result we need is the trace of A. We might guess that the appropriate expression is

$$\mathrm{Tr}\,\hat{A} = \sum_{\bar{c},c}\langle\bar{c}|\hat{A}|c\rangle.$$

This is almost right, but in fact it turns out that the anticommuting properties of the Grassmans force us to introduce a minus sign into this expression:

$$\mathrm{Tr}\,\hat{A} = \sum_{\bar{c},c}\langle-\bar{c}|\hat{A}|c\rangle = \int d\bar{c}dce^{-\bar{c}c}\langle-\bar{c}|\hat{A}|c\rangle, \tag{12.110}$$

Grassman trace formula

which, we shall shortly see, gives rise to the antisymmetric boundary conditions of fermionic fields. To prove the above result, we rewrite (12.105) as

$$\delta_{nm} = \langle n|m\rangle = \int d\bar{c}dce^{-\bar{c}c}\langle-\bar{c}|m\rangle\langle n|c\rangle, \tag{12.111}$$

where the minus sign arises from anticommuting c and $\bar{c}$. We can now rewrite the trace as

$$\begin{aligned}\mathrm{Tr}\,A &= \sum_{n,m}\langle m|A|n\rangle\delta_{nm} \\ &= \sum_{n,m}\int d\bar{c}dce^{-\bar{c}c}\langle-\bar{c}|m\rangle\langle m|A|n\rangle\langle n|c\rangle \\ &= \int d\bar{c}dce^{-\bar{c}c}\langle-\bar{c}|\hat{A}|c\rangle.\end{aligned} \tag{12.112}$$

We shall make extensive use of the completeness and trace formulae (12.106) and (12.110) in developing the path integral. Both expressions are simply generalized to many fields c_j

so that (12.194) factorizes into a radial and an angular integral:

$$I_{nm} = \frac{1}{\sqrt{n!\,m!}} \int \frac{d\bar{b}db}{2\pi i} \bar{b}^n \bar{b}^m e^{-\bar{b}b} = \frac{1}{\sqrt{n!\,m!}} \int_0^\infty 2rdr r^{n+m} e^{-r^2} \times \overbrace{\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi(n-m)}}^{\delta mn}, \tag{12.195}$$

where we have substituted $\langle n|b\rangle = \frac{1}{\sqrt{n!}} b^n$ and $\langle \bar{b}|m\rangle = \frac{1}{\sqrt{m!}} \bar{b}^m$. The angular integral vanishes unless $n = m$. Changing variables $r^2 \to x$, $2rdr = dx$, in the first integral, we then obtain

$$I_{nm} = \frac{\delta_{nm}}{n!} \int_0^\infty dx\, x^n e^{-x} = \delta_{nm}, \tag{12.196}$$

proving the orthogonality relation. Now since $\delta_{nm} = \langle n|m\rangle$, we can write the orthogonality relation (12.194) as

$$\langle n|m\rangle = \int \frac{d\bar{b}db}{2\pi i} e^{-\bar{b}b} \langle n|b\rangle\langle \bar{b}|m\rangle = \langle n| \left(\int \frac{d\bar{b}db}{2\pi i} e^{-\bar{b}b} |b\rangle\langle \bar{b}| \right) |m\rangle.$$

Since this holds for all states $|n\rangle$ and $|m\rangle$, it follows that the quantity in brackets is the unit operator:

$$\hat{1} = \int \frac{d\bar{b}db}{2\pi i} e^{-\bar{b}b} |b\rangle\langle \bar{b}| = \int \frac{d\bar{b}db}{2\pi i} \frac{|b\rangle\langle \bar{b}|}{\langle \bar{b}|b\rangle} \equiv \sum_{\bar{b},b} |b\rangle\langle \bar{b}|. \text{ Completeness relation} \tag{12.197}$$

Appendix 12B Grassman differentiation and integration

Differentiation is defined to have the normal linear properties of the differential operator. We denote

$$\partial_c \equiv \frac{\partial}{\partial c}, \quad \partial_{\bar{c}} \equiv \frac{\partial}{\partial \bar{c}}, \tag{12.198}$$

so that

$$\partial_c c = \partial_{\bar{c}} \bar{c} = 1. \tag{12.199}$$

If we have a function

$$f(\bar{c}, c) = f_0 + \bar{f}_1 c + \bar{c} f_1 + f_{12} \bar{c} c, \tag{12.200}$$

then differentiation from the left-hand side gives

$$\begin{aligned} \partial_c f &= \tilde{f}_1 - f_{12}\bar{c} \\ \partial_{\bar{c}} f &= f_1 + f_{12} c, \end{aligned} \tag{12.201}$$

where the minus sign in the first expression occurs because the $\bar{\partial}$ operator must anticommute with c. But how do we define integration? This proves to be much easier for Grassman variables than for regular c-numbers. The great sparseness of the space of functions dramatically restricts the number of linear operations we can apply to functions, forcing differentiation and integration to become the *same* operation:

$$\int dc \equiv \partial_c, \qquad \int d\bar{c} \equiv \partial_{\bar{c}}. \tag{12.202}$$

In other words,

$$\int d\bar{c}\bar{c} = 1, \qquad \int dcc = 1, \qquad \int d\bar{c} = \int dc = 0. \tag{12.203}$$

Appendix 12C Grassman calculus: change of variables

Suppose we change variables, writing

$$\begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \tag{12.204}$$

where A is a c-number matrix. Then we would like to know how to evaluate the Jacobian for this transformation, which is defined so that

$$\int dc_1 \cdots dc_r \, [\ldots] = \int J\left(\frac{c_1 \cdots c}{\xi_1 \cdots \xi_r}\right) d\xi_1 \cdots d\xi_r \, [\ldots]\,. \tag{12.205}$$

Now since integration and differentiation are identical for Grassman variables, we can evaluate the fermionic Jacobian using the chain rule for differentiation, as follows:

$$\begin{aligned} \int dc_1 \cdots dc_r \, [\ldots] &= \frac{\partial^r}{\partial c_1 \cdots \partial c_r} \, [\ldots] \\ &= \sum_P \left(\frac{\partial \xi_{P_1}}{\partial c_1} \cdots \frac{\partial \xi_{P_r}}{\partial c_r} \right) \frac{\partial^r}{\partial \xi_{P_1} \cdots \partial \xi_{P_r}} \, [\ldots]\,, \end{aligned} \tag{12.206}$$

where $P = \begin{pmatrix} 1 & \cdots & r \\ P_1 & \cdots & P_r \end{pmatrix}$ is a permutation of the sequence $(1 \cdots r)$. But we can order the differentiation in the second term, picking up a factor $(-1)^P$, where P is the signature of the permutation, to obtain

$$\begin{aligned} \int dc_1 \cdots dc_r \, [\ldots] &= \sum_P (-1)^P \left(\frac{\partial \xi_{P_1}}{\partial c_1} \cdots \frac{\partial \xi_{P_r}}{\partial c_r} \right) \frac{\partial^r}{\partial \xi_1 \cdots \partial \xi_r} \, [\ldots] \\ &= \det[A^{-1}] \frac{\partial^r}{\partial \xi_1 \cdots \partial \xi_r} \, [\ldots] \\ &= \int \det[A^{-1}] d\xi_1 \cdots d\xi_r \, [\ldots]\,, \end{aligned} \tag{12.207}$$

where we have recognized the prefactor as the determinant of the inverse transformation $\xi = \underline{A}^{-1}c$. From this result, we can read off the Jacobian of the transformation as

$$J\left(\frac{c_1 \ldots c_r}{\xi_1 \ldots \xi_r}\right) = \det[A]^{-1} = \left|\frac{\partial(c_1, \ldots c_r)}{\partial(\xi_1, \ldots \xi_r)}\right|^{-1}, \tag{12.208}$$

which is precisely the inverse of the bosonic Jacobian. This has important implications for supersymmetric field theories, where the Jacobians of the bosons and fermions precisely cancel. For our purposes, however, the most important point is that, for a unitary transformation, the Jacobian is unity.

Appendix 12D Grassman calculus: Gaussian integrals

The basic Gaussian integral is simply

$$\int d\bar{c}dc e^{-a\bar{c}c} = \int d\bar{c}dc(1 - a\bar{c}c) = a. \tag{12.209}$$

If we now introduce a set of N variables, then

$$\int \prod_j d\bar{c}_j dc_j \exp - \left[\sum_j a_j \bar{c}_j c_j\right] = \prod_j a_j. \tag{12.210}$$

Suppose we now carry out a unitary transformation, for which the Jacobian is unity. Then, since

$$c = U\xi, \qquad \bar{c} = \bar{\xi}U^\dagger,$$

the integral becomes

$$\int \prod_j d\bar{\xi}_j d\xi_j \exp[-\bar{\xi} \cdot A \cdot \xi] = \prod_j a_j,$$

where $A_{ij} = \sum_l U^\dagger_{il} a_l U_{lj}$ is the matrix with eigenvalues a_l. It follows that

$$\int \prod_j d\bar{\xi}_j d\xi_j \exp[-\bar{\xi} \cdot A \cdot \xi] = \det A. \tag{12.211}$$

Finally, by shifting the variables $\xi \to \xi + A^{-1}j$, where j is an arbitrary vector, we find that

$$Z[j] = \int \prod_j d\bar{\xi}_j d\xi_j \exp[-(\bar{\xi} \cdot A \cdot \xi + \bar{j} \cdot \xi + \bar{\xi} \cdot j)] = \det A \exp[\bar{j} \cdot A^{-1} \cdot j]. \tag{12.212}$$

This is the basic Gaussian integral for Grassman variables. Notice that, using the result $\ln \det A = \mathrm{Tr} \ln A$, it is possible to take the logarithm of both sides to obtain

$$S[j] = -\ln Z[j] = -\mathrm{Tr} \ln A - \bar{j} \cdot A^{-1} \cdot j. \tag{12.213}$$

The main use of this integral is for evaluating the path integral for free-field theories. In this case, the matrix $A \to -G^{-1}$ becomes the inverse propagator for the fermions, and $\xi_n \to \psi(i\omega_n)$ is the Fourier component of the Fermi field at Matsubara frequency $i\omega_n$.