THE QUANTUM ELECTRON PLASMA (RPA)

$$\frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\nabla\times A\right)^2 =$$

$$\mathcal{L}_{EM} = T - V$$

$$\frac{\epsilon_0 E^2}{2} = \left(-\frac{\partial A}{\partial t} - \nabla\phi\right)^2 \frac{\epsilon_0}{2} = \frac{1}{2\mu_0}\left(\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\left(\frac{\phi}{c}\right)\right)^2$$

cgs $\epsilon_0 \to 1/4\pi$, $\mu_0 \to 4\pi$

$$\frac{1}{8\pi}\left(\frac{E^2}{c^2} - B^2\right)$$

$$S_Q = \frac{1}{\hbar}\int d^4x \left[\frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0}\right] \longrightarrow \text{Photons!}$$

Coulomb Gauge $\nabla . A = 0$ $\quad -\epsilon_0 \left[\phi \frac{\partial}{\partial t}(\nabla . A)\right] = 0$

$$S_Q = \frac{1}{\hbar}\int \left(\frac{\partial A}{\partial t}\right)^2 \frac{\epsilon_0}{2} + \frac{\epsilon_0 (\nabla\phi)^2}{2} - (\nabla\times A)^2 - e\phi\rho$$

$$S_T = \sim\int d^4x \left[\frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0}\right]$$

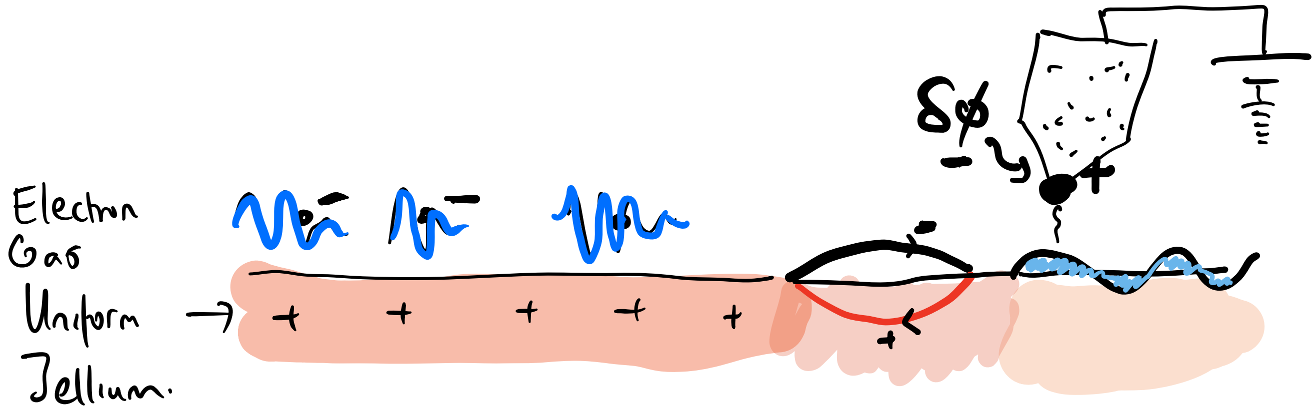
$$\boxed{S = \int d^3x\, d\tau \left[\psi^\dagger\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu + e\,\phi(x)\right)\psi - \frac{\epsilon_0}{2}(\nabla\phi)^2\right]}$$

Coulomb Gauge

$$e\phi \to \phi$$

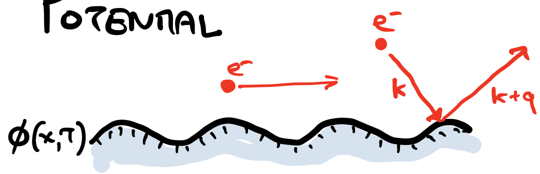
$$S = \int d^3x\, d\tau \left[\psi^\dagger\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu + \phi\right)\psi - \frac{\epsilon_0 e^2}{2}(\nabla\phi)^2\right]$$

$$-G_0^{-1} = \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right) \qquad -G^{-1} = -G_0^{-1} + \phi$$



$$\boxed{S = \int d^3x\, d\tau \left[\psi^{+} - G_0^{-1} \psi + \phi\left(\psi^{+}\psi - \rho_{+}\right) - \frac{e^2\epsilon_0}{2}\left(\nabla\phi\right)^2 \right]}$$

Feynman Diagrams: Fermions in a Varying Potential



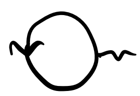
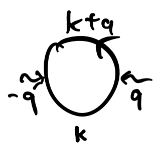
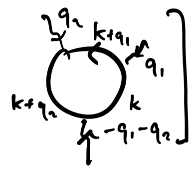
$$F = -T\,\mathrm{Tr}\ln\left[-G_0^{-1} + \phi\right]$$

$$= -T\,\mathrm{Tr}\ln\left(-G_0^{-1}\right) - T\,\mathrm{Tr}\left[1 - G_0\phi\right]$$

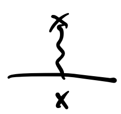
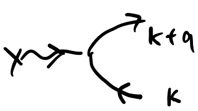
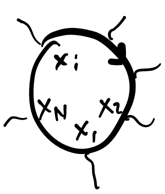
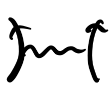
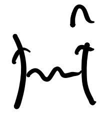
$$= F_0 + T\,\mathrm{Tr}\left[G_0\phi + \frac{1}{2}\left(G_0\phi\right)^2 + \frac{1}{3}\left(G_0\phi\right)^3 + \dots\right]$$

$$= F_0 - T\left[-\int d^4x\, G_0(x^-, x)\phi(x) - \frac{1}{2}\int d^4x\, d^4y\, \phi(x) G_0(x-y)\phi(y) G_0(y-x) - \dots\right]$$

Fermion loop (NOTE DOUBLE NEGATIVE).

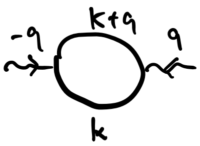
$S = \beta F_0 - \Big[$  $+$  $+$  $\Big]$

$$Z = e^{-S} = \sum\left(\begin{array}{l}\text{closed} \\ \text{unlinked} \\ \text{diagrams}\end{array}\right)$$

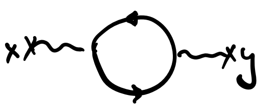
	REAL SPACE	MOMENTUM SPACE
$x' \longrightarrow x$	$G_0(x-x') = -\langle T\,\psi(x)\psi^\dagger(x')\rangle_{H_0}$	$G_0(k) = \dfrac{1}{i\omega_n - \epsilon_k}$
	$\phi(x)$	 $\phi(q)$; $\phi_{-q} = \phi_q^*$
	$\prod_{i=1}^{N}\int d^4x_i$ all intermediate co-ordinates	$\prod^{\text{loops}} \frac{1}{V\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}$ All loop momenta.
	$(-1)^F$ $F = \#$ Fermion loops.	$(-1)^F$ Fermion loops.
	$\frac{1}{n}$ symmetry factor	$\frac{1}{n}$ symmetry factor.
	$-V(x-y)$ interaction	 $-V(q)$

$$\frac{\Delta S}{\beta V} = \frac{\Delta F}{V} = -\frac{1}{\beta V}\left[\sum \text{closed link diagrams in real space}\right]$$

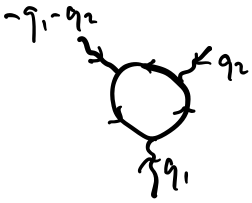
$$= -\sum\left[\text{closed link diagrams in momentum space.}\right]$$



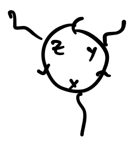
$$= (-)^2 \frac{N}{2\beta V}\sum \phi_{-q}\,\phi_q\, G(k+q)\, G(k)$$



$$= -\frac{N}{2\beta V}\int d^4x\, d^4y\;\; \phi(x)\,\phi(y)\, G(x-y)\, G(y-x)$$

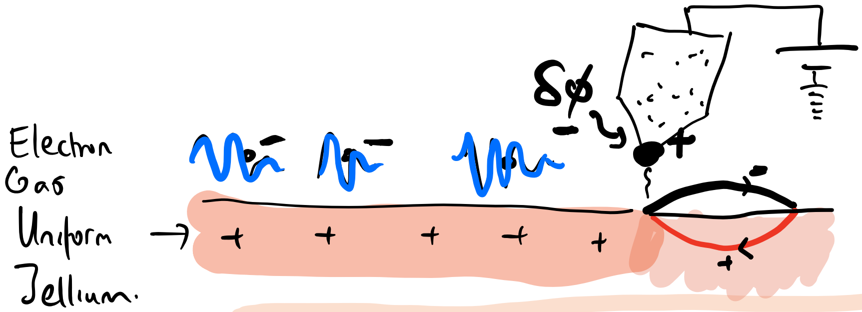


$$= -\frac{N}{3\beta V}\sum \phi_{-q_1-q_2}\,\phi_{q_1}\,\phi_{q_2}\; G(k+q_2)\,G(k+q_1)\,G(k)$$



$$= -\frac{N}{3\beta V}\int d^4x\, d^4y\, d^4z\;\phi(x)\,\phi(y)\,\phi(z)\;G(x-z)\,G(z-y)\,G(y-x)$$

"Dolphins leaping out of the fermi sea"



$$\langle \psi^\dagger(x)\psi(x)\rangle = \rho_0$$

$$\boxed{S = \int d^4x \left\{ \bar{\psi}_\sigma\left(-\mathcal{G}_0^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) - \frac{\epsilon_0}{2e^2}\left(\nabla\phi\right)^2 \right\}}$$

Make saddle point exact $\sigma = 1, \dots N$ $\quad \epsilon_0 = \tilde{\epsilon}_0$

Repulsive interaction $\left(\frac{e^2}{\epsilon_0 q^2}\right)$

$$S = \int d^4x \left\{ \bar{\psi}_\sigma\left(-\mathcal{G}_0^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) \underbrace{- \frac{\epsilon_0}{2e^2}\left(\nabla\phi\right)^2}_{-\frac{\epsilon_0 q^2}{2e^2}|\phi_q|^2} \right\}$$

" $|\delta\phi_q|^2 = \frac{-e^2}{\epsilon_0 q^2}$ "

Repulsive

$$\frac{e^2}{\epsilon_0} = \frac{\tilde{e}^2}{\epsilon_0 N} \qquad \frac{e^2}{4\pi\epsilon_0 r} = \left(\frac{\tilde{e}^2}{4\pi\epsilon_0 r\, N}\right)$$

$$\boxed{S = \int d^4x \left\{ \bar{\psi}_\sigma\left(-\mathcal{G}^{-1}\right)\psi_\sigma + \phi\left(\bar{\psi}_\sigma\psi_\sigma - \rho_0\right) - \frac{N\epsilon_0}{2\tilde{e}^2}\left(\nabla\phi\right)^2 \right\}}$$

$$Z = \int \mathcal{D}[\phi]\, e^{\frac{N\epsilon_0}{e^2 2}\int d\tau\, d^3x\, (\nabla\phi)^2} \int \mathcal{D}[\bar\psi,\psi]\, e^{-N\int d^4x \left[\bar\psi\left(-G_0^{-1} + \phi\right)\psi - \phi\rho_0\right]}$$

tue backgd

$$= \int \mathcal{D}[\phi]\, e^{-S_\phi}\, e^{N \operatorname{Tr}\ln\left(-G_0^{-1} + \phi\right)}$$

$$= \int \mathcal{D}[\phi]\, e^{-\left[S_\phi - \operatorname{Tr}\ln\left(-G_0^{-1} + \phi\right)\right]N}$$

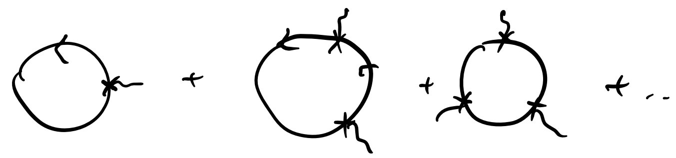
$$= \int \mathcal{D}[\phi]\, e^{-S_{Eff}[\phi]}$$

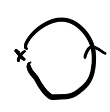
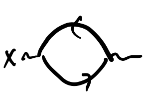
$$\frac{\beta F}{N} = \frac{S_{Eff}[\phi]}{N} = -\operatorname{Tr}\ln\left[-G_0^{-1} + \phi\right] - \int d^4x \left[\frac{\epsilon_0}{2e^2}(\nabla\phi)^2 + \phi\tilde\rho_0\right]$$

$$= -\operatorname{Tr}\ln\left[-G_0^{-1}\left(1 - G_0\phi\right)\right] - S_\phi$$

$$= -\operatorname{Tr}\ln\left(-G_0^{-1}\right) + \int d^4x\, \phi\left(G_0 - \rho_0\right) + \frac{1}{2}\operatorname{Tr}\left(G_0\phi G_0\phi\right) + O(\phi^3)$$

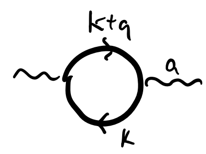
$$- \int d^4x \left[\frac{\epsilon_0}{2e^2}(\nabla\phi)^2\right]N$$

$S_{EFF} = S_0 - \Big[$  $+ \dots \Big] - \phi\rho_0$

$= S_0 - \int d^4x \Big[\rho_0 +$  $\Big]\phi(x) - \int \frac{\phi(x)}{2} \Big[x$  $x' + \frac{\epsilon_0}{e^2}\nabla^2 \Big]\phi(x')\, d^3x\, d^3x'$

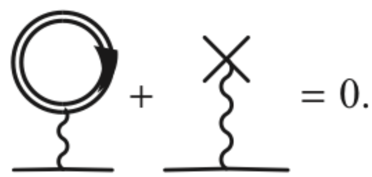
$-G(0^-)N$

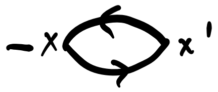
$+ O(\phi^3)$

$\simeq N S_0 - \sum_k \frac{1}{2}\Big[$  $+ \frac{N q^2}{e^2} \Big]|\phi_q|^2 + O(\phi^3)$

$$\frac{\delta S_{EFF}}{\delta \phi} = 0 \quad \Rightarrow \quad N G_0(\vec{x}^-, x) = \rho_0 = -\langle T \psi_\sigma(x^-) \psi^\dagger_\sigma(x) \rangle = \langle \psi^\dagger_\sigma(x) \psi_\sigma(x) \rangle$$

$\boxed{\text{NEUTRALITY}}$



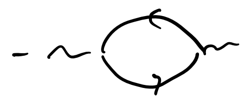


$$\frac{\delta^2 S}{e^2 \delta\phi(x) \delta\phi(x')} = \left[\overbrace{N\, G(x-x') G(x-x')} - \frac{\epsilon_0 N \nabla^2}{\tilde{e}^2} \right]$$

$$G(x) = \frac{T}{V} \sum G(k) e^{ikx}$$

$$\phi(x) = \frac{1}{\sqrt{\beta V}} \sum \phi(k) e^{ikx}$$

$$\phi(x)\, G(x-x')\, G(x'-x)\, \phi(x')$$



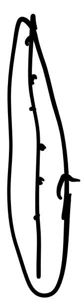
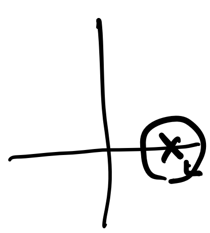
$$S_{QUAN} = \frac{N}{2} \sum \left[G(k+q) G(k) + \frac{\epsilon_0}{\tilde{e}^2} q^2 \right] |\phi_q|^2$$

$$-\frac{1}{N\beta V} \frac{\delta^2 S}{\delta \phi_q^* \delta \phi_q} = \tilde{\chi}(q) = -\frac{1}{V\beta} \sum G(k+q) G(k)$$

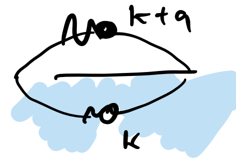
$$\tilde{\chi}(\vec{q}, i\nu_n) = -\frac{1}{V} \sum_k T \sum \frac{1}{i\omega_n + i\nu_n - \epsilon_{k+q}} \frac{1}{i\omega_n - \epsilon_k}$$

$$= \frac{1}{V} \sum_k \oint \frac{f(z)\, dz}{2\pi i} \frac{1}{z + i\nu_n - \epsilon_{k+q}} \frac{1}{z - \epsilon_k}$$

poles on imaginary axis

 = 

$\epsilon_{k+q} - i\nu_n$



$$= -\frac{1}{V}\sum \oint \frac{f(z)\,dz}{2\pi i} \; \frac{1}{z-(\epsilon_{k+q}-i\nu_n)} \; \frac{1}{z-\epsilon_{\vec{k}}}$$

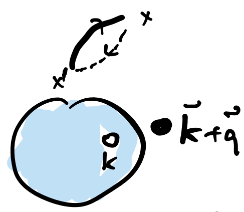
Poles at $z = \epsilon_k,\ \epsilon_{k+q} - i\nu_n$

$$= -\frac{1}{V}\sum_{\vec{k}} \left[\frac{f(\epsilon_{k+q}-i\nu_n)}{\epsilon_{k+q}-\epsilon_k - i\nu_n} + \frac{f(\epsilon_k)}{\epsilon_k - \epsilon_{k+q} + i\nu_n} \right.$$

$$\tilde{\chi}(\vec{q}, i\nu_n) = \frac{1}{V}\sum_k \left[\frac{f_{k+q} - f_k}{i\nu_n - (\epsilon_{k+q}-\epsilon_k)} \right] = \int \frac{d^3k}{(2\pi)^3} \left(\frac{f_{k+q}-f_k}{i\nu_n - (\epsilon_{k+q}-\epsilon_k)} \right)$$

$$\chi(q,z) = Ne^2 \int \frac{d^3k}{(2\pi)^3} \frac{f_{k+q}-f_k}{z-(\epsilon_{k+q}-\epsilon_k)}$$

$\boxed{\text{Lindhardt Function}}$ "Charge susceptibility"

$+1$ 

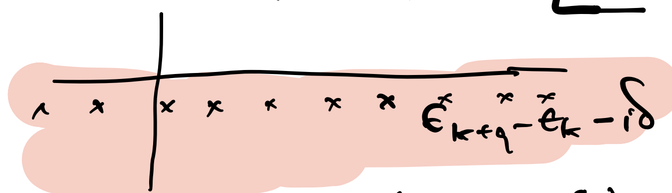
$$\frac{1}{\omega - \omega_q}$$

Retarded. response function.
Particle-hole excitation.

$$\tilde{\chi}_R(\vec{q},\nu) = \chi(q,z)\Big|_{z=\nu+i\delta}$$

$$\chi_A = \chi(q,t)\Big|_{z=\nu-i\delta}$$

LRZ



$$\left(\frac{1}{\omega-\omega_q} - \frac{1}{\omega+\omega_{-q}}\right) = \left(\frac{2\omega_q}{\omega^2-\omega_q^2}\right)$$

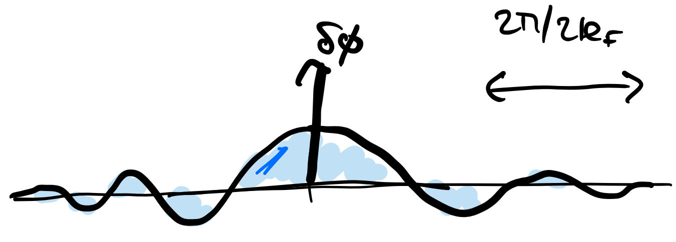
$$\delta\rho(q) = \chi(q)\,\phi(q).$$

$$\chi(x) = \int \frac{d^4q}{(2\pi)^4} \chi(q) e^{iqx}$$

$$\delta\rho(x,t) = \int d^4x'\, \chi(x-x')\,\phi(x')$$

$$\delta\phi = \phi\,\delta^3(\vec{x})$$

$$\delta\rho(\vec{x}) = \int d^3x'\, \chi(\vec{x}\quad, \omega=0)\,\phi_0$$



Friedel oscillations of charge.