INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2015

Questions 5: Due Fri 13th Nov

1. Consider a gas of particles with interaction

\[ \hat{V} = \frac{1}{2} \sum_{kk'q} V_{qq'}^{c_k c_{k'} c_q c_{q'}} \]

(a) Let \( |\phi_0\rangle = \prod_{|k|<k_F,\sigma} c_k^{\dagger} |0\rangle \) represent the filled Fermi sea, i.e. the ground state of the non-interacting problem. Use Wick’s theorem to evaluate an expression for the expectation value of the interaction energy \( \langle \phi_0 | \hat{V} | \phi_0 \rangle \) in the non-interacting ground state. Give a physical interpretation of the two terms that arise and draw the corresponding Feynman diagrams. You may use the result

\[ \langle \phi | c_k^{\dagger} c_{k'} \phi \rangle = \delta_{kk'} \delta_{ss'} \theta(k_F - k) \]  

(b) Draw all the Feynman diagrams corresponding to the second order corrections to the ground-state energy.

(c) Discuss how to convert one or more of the above Feynman diagrams into a mathematical expression, using the Feynman rules.

2. The separation of electrons \( R_e \) in a Fermi gas is defined by

\[ \frac{4\pi R_e^3}{3} = \rho^{-1} \]

where \( \rho \) is the density of electrons. The dimensionless separation \( r_s \) is defined as \( r_s = R_e/a \) where \( a = \frac{\hbar^2}{me^2} \) is the Bohr radius.

(a) Show that the Fermi wavevector is given by

\[ k_F = \frac{1}{\alpha r_s a} \]

where \( \alpha = \left( \frac{4}{\pi^2} \right)^{\frac{1}{3}} \approx 0.521 \).

(b) Consider an electron plasma where the background charge density precisely cancels the charge density of the plasma. Show that the ground-state energy to leading order in the strength of the Coulomb interaction is given by

\[ \frac{E}{\rho V} = \frac{3}{5} \frac{R_Y}{\alpha^2 r_s^2} - \frac{3}{2\pi} \frac{R_Y}{\alpha r_s} \]

\[ = \left( \frac{2.21}{r_s^2} - \frac{0.912}{r_s} \right) R_Y \]  

where \( R_Y = \frac{\hbar^2}{2me^2} \) is the Rydberg energy. (Hint - in the electron gas with a constant charge background, the Hartree part of the energy vanishes. The Fock part is the second term in this expression. You may find it useful to use the integral

\[ \int_0^1 dx \int_0^1 dy \ln \left| \frac{x+y}{x-y} \right| = \frac{1}{2} \]
(c) When can the interaction effects be ignored relative the kinetic energy?

3. Consider a Fermi gas containing Fermions with spin degeneracy \( n_s = 2S + 1 > 2 \), and total density \( n_s \rho \), where \( \rho \) is the density per spin component. Nucleons can be considered as the special case where \( n_s = 4 \), corresponding to spin and isospin quantum number.

(a) Suppose the fermions interact via a repulsive three body interaction:

\[
V(r_i, r_j, r_k) = \beta \delta^{(3)}(r_i - r_j)\delta^{(3)}(r_j - r_k),
\]

Write this interaction in second-quantized form.

(b) Invent a symbol for a three body interaction and write down the Feynman rules.

(c) Use your rules to calculate the interaction energy per unit volume, \( V(n_s, \rho) \) to leading order in \( \alpha \). What happens when \( n_s = 1 \) or \( n_s = 2 \)?

(d) If we neglect Coulomb interactions, why is the case \( n_s = 4 \) relevant to nuclear matter?