GRADUATE QUANTUM MECHANICS: 502 Spring 2002

Solution to Assignment 5

1. In two dimensions, the number of electrons in a volume of phase space $L^2 d^2k$ is given by

$$dN = L^2 \frac{d^2k}{(2\pi)^2}.$$

Writing $d^2k = kdkd\phi$ in polar co-ordinates, the number of electrons per unit energy is then given by

$$dN = L^2 \frac{1}{(2\pi)^2} k(dk/dE)dEd\phi = \rho(E)dEd\phi$$

Now $E = \hbar^2k^2/2m$, so that $dk/dE = m/(\hbar^2 k)$ and so the density of states per unit energy, per unit angle is independent of energy and angle, and given by

$$\rho(E,\phi) = \rho = L^2 \frac{m}{h^2}.$$

where $h = 2\pi\hbar$ is Planck's constant.

2. Solution to Sakurai problem 38, Ch. 2.

From the Golden rule, we know that the rate of transition from initial to final state is

$$dw_{j\to[f]} = \frac{2\pi}{\hbar} \sum_{f\in d\Omega} \frac{e^2}{m^2} A_o^2 |\langle f|e^{i\vec{q}\cdot\vec{x}}\hat{\epsilon}\cdot\hat{p}|j\rangle|^2 \delta(E_f - E_j - \hbar\omega).$$

Here A_o is the amplitude of the vector potential, which has polarization vector $\hat{\epsilon}$, and is given by $\vec{A}(\vec{x},t) = 2A_o\cos(\vec{q}\cdot\vec{x} - \omega t)\hat{\epsilon}\mathrm{r}$. We can obtain the differential scattering cross-section by normalizing this with respect to the energy flux of the radiation field, given by

$$S = \frac{1}{2\pi}\left(\frac{4\pi}{\mu_0}\right)\frac{\omega^2}{c}A_o^2.$$

We then obtain

$$d\sigma = \frac{dw_{j\to[f]}}{S} = \frac{4\pi^2\hbar}{m^2\omega}\alpha \sum_{f\in d\Omega} |\langle f|e^{i\vec{q}\cdot\vec{x}}\hat{\epsilon}\cdot\hat{p}|j\rangle|^2 \delta(E_f - E_j - \hbar\omega)$$

where $\alpha = e_{cgs}^2/\hbar c$ is the fine structure constant. The sum over final states $|f\rangle$is given by

$$\begin{aligned}
\sum_{f\in d\Omega}\{\ldots\} &= L^3 \int \frac{k_f^2 dk_f}{(2\pi)^3} d\Omega\{\ldots\}\delta(E_f - E_j - \hbar\omega) \\
&= L^3 \int \frac{mk_f}{(2\pi)^3\hbar^2} d\Omega dE_f\{\ldots\}\delta(E_f - E_j - \hbar\omega). \\
&= L^3 \frac{mk_f}{(2\pi)^3\hbar^2}\{\ldots\}d\Omega\bigg|_{E_f = E_i + \hbar\omega} \qquad (1)
\end{aligned}$$

The matrix element can be evaluated as

$$|\langle f|e^{i\vec{q}\cdot\vec{x}}\hat{\epsilon}\cdot\hat{p}|j\rangle|^2 = \frac{\hbar}{L^{\frac{3}{2}}} \int d^3x e^{i(\vec{q}-\vec{k}_f)\cdot\vec{x}}(-i\hbar\hat{\epsilon}\cdot\vec{\nabla})\psi_j(x)$$

$$
\begin{aligned}
&= \frac{\hbar}{L^{\frac{3}{2}}}\int d^3x\psi_j(x)(i\hbar\hat{\epsilon}\cdot\vec{\nabla})[e^{i(\vec{q}-\vec{k}_f)\cdot\vec{x}}] \\
&= \frac{\hbar}{L^{\frac{3}{2}}}(\hat{\epsilon}\cdot\vec{Q})\int d^3x\psi_j(x)e^{-i\vec{Q}\cdot\vec{x}} \\
&= \frac{\hbar}{L^{3/2}}(\hat{\epsilon}\cdot\vec{Q})\phi_j(\vec{Q}), \qquad (2)
\end{aligned}
$$

where $\vec{Q} = \vec{k}_f - \vec{q}$ and

$$
\phi_j(\vec{Q}) = \int d^3x\psi_j(x)e^{-i\vec{Q}\cdot\vec{x}}
$$

is the Fourier transform of the initial state. Putting these two results together, we obtain the differential scattering cross section

$$
\begin{aligned}
d\sigma &= \frac{4\pi^2\hbar}{m^2\omega}\alpha\int(\hbar\hat{\epsilon}\cdot\vec{Q})^2|\phi_j(\vec{Q})|^2\frac{m_{k_f}}{(2\pi)^3\hbar^2}dE\delta(E_f-E_j-\hbar\omega)d\Omega \\
&= \frac{e_{cgs}^2}{2\pi m\omega c}(\hat{\epsilon}\cdot\vec{Q})^2k_f|\phi_j(\vec{Q})|^2 \qquad (3)
\end{aligned}
$$

To finish the job, we need to compute the Fourier transform of the initial state, which is the ground state of the 3D harmonic oscillator

$$
\psi_j(\vec{x}) = \frac{1}{(2\pi\Delta x)^{\frac{3}{2}}}e^{-(x^2+y^2+z^2)/4(\Delta x)^2},
$$

where $(\Delta x)^2 = \frac{\hbar}{2m\omega}$. Since the Fourier transform of a Gaussian is a Gaussian,

$$
\int dxe^{iqx}e^{-x^2/2\sigma^2} = \sqrt{2\pi\sigma^2}e^{-q^2\sigma^2/2},
$$

carrying out the 3D Fourier transform in Cartesian co-ordinates, $d^3x = dxdydz$, we obtain

$$
\phi_j(Q) = (2\Delta x)^{\frac{3}{2}}e^{-Q^2\Delta x^2},
$$

so that the differential scattering cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{e_{cgs}^2}{2\pi m\omega c}(\hat{\epsilon}\cdot\vec{Q})^2k_f\left(\frac{2\hbar}{m\omega}\right)^{\frac{3}{2}} exp\left(-\frac{\hbar Q^2}{m\omega}\right).
$$

It is always good to try to simplify expressions like this. First note that the energy of the final state is given by

$$
E_f = \frac{\hbar^2k_f^2}{2m} = \hbar\omega + 3\frac{\hbar\omega_o}{2}
$$

where ω_o is the frequency of the harmonic oscillator. Now in practice, since the velocity of the photon is much faster than that of the outgoing electron (for comparable energies), the momentum of the outgoing electron must be much greater than the momentum of the incoming photon, i.e $k_f >> q$ and $\vec{Q} \approx \vec{k}_f$, so that $\frac{\hbar^2Q^2}{2m} \approx E_f$. Furthermore, if we take the direction of the incoming photon to be along the z axis and the polarization vector of the incoming radiation to lie along the x direction, then $\hat{\epsilon}\cdot\vec{Q} = k_{fx} = k_f\sin\theta\cos\phi$, so that the differential scattering cross-section can be written in the compact form

$$
\frac{d\sigma}{d\Omega} = \sigma_o\cos^2\phi\sin^2\theta
$$

where

$$
\sigma_o = \frac{e_{cgs}^2}{2\pi m\omega c}\left(\frac{4E_f}{\hbar\omega}\right)^{\frac{3}{2}}\exp\left(-\frac{2E_f}{\hbar\omega}\right).
$$

3. (a) The energy of the n-th one-particle state in the harmonic oscillator potential is

$$\hbar\omega(n - 1/2), \tag{4}$$

where the "$-1/2$", rather than $+1/2$ is there because we are $n = 1, 2, 3 \ldots$, since we start counting from the first state. For a spin 1/2 particle, each one-particle state is doubly occupied, with a spin up and a spin down particle. Assuming that the number of particles N is even, the first $N/2$ levels are then filled in the ground-state, so that the ground-state energy is

$$E_g = 2 \sum_{n=1,N/2} \hbar\omega(n - 1/2) = \frac{1}{2}\hbar\omega N(N+1) \tag{5}$$

For a system with a discrete energy level structure, the position of the Fermi energy is actually a bit tricky. If we take the Fermi energy to be the zero temperature limit of the chemical potential, then this will lie half way between the last occupied leve and the first occupied level. The Fermi energy is then

$$\epsilon_F = \hbar\omega(N-1)/2 + \hbar\omega/2 = \frac{N}{2}\hbar\omega \tag{6}$$

(b) On reflection, I was not sure what Sakurai intended by the second half of this question. In the large N limit, we can take the leading order N dependence, so that the ground-state and Fermi energies are given by

$$E_g = \frac{1}{2}\hbar\omega N^2, \qquad \epsilon_F = \frac{N}{2}\hbar\omega \tag{7}$$

4. The list is endless!- Here are my three suggestions:

- Our ability to stand on the floor. The exclusion principle is responsible for the strong repulsive force between atoms which come into close contact, and without this effect we would literally fall through the floor! When atoms try to occupy the same volume, the Exclusion principle forces the electrons in the atom to move to higher orbitals, causing a rapid rise in the energy of the atom. If there were no exclusion principle, atoms would be able to pass through one-another just like classical galaxies.
- Chemistry! The diversity in the properties of atoms, and the richness of chemistry which results, is all a consequence of the exclusion principle. If electrons were bosons, then they would all occupy the lowest $n = 1$ orbital. The physics and symmetries of all atoms would then be remarkably similar and dull.
- Electrical properties of metals and semiconductors rely on the exclusion principle. The exclusion principle causes electrons to occupy states up to a finite Fermi energy. In a typical metal, the Fermi energy is of order $10,000K$, far in excess of the ambient temperature. The robust, high conductivity of metals, together with their tiny electronic specific heat, is all a consequence of this high Fermi energy.

5. This was a tricky little question. Since the system has an axial symmetry about the z-axis, the $z-$ component of the angular moment is conserved. Suppose that the state of the system has $J_z = \hbar m_z$, then under a 120^0 rotation, the wavefunction of the system transforms as follows:

$$|m_z\rangle \to e^{-i\frac{2\pi}{3}J_z/\hbar}|m_z\rangle = e^{-im_z\frac{2\pi}{3}}|m_z\rangle \tag{8}$$

Now we can achieve a rotation through 120^0 by carrying out three exchanges of the particles around the equilateral triangle. Since the particles are bosons, each exchange leaves the wavefunction unchanged, thus we must have

$$e^{-im_z\frac{2\pi}{3}}|m_z\rangle = |m_z\rangle \tag{9}$$

which is only possible if m_z is a multiple of three.