GRADUATE QUANTUM MECHANICS: 502 Spring 2002

Solutions to assignment 3.

1. (Solution to Sakurai, problem 4, ch 5) The isotropic Harmonic oscillator in two dimensions

$$H_o = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2). \tag{1}$$

can be re-written as

$$H_o = \hbar\omega(a^\dagger a + b^\dagger b + 1) \tag{2}$$

Where

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p_x}{m\omega}\right) \\ b &= \sqrt{\frac{m\omega}{2\hbar}}\left(y + i\frac{p_y}{m\omega}\right) \end{aligned} \tag{3}$$

are the annihilation operators for modes in the x and y directions, respectively. The eigenvalues of the number operators $n_a = a^\dagger a$ and $n_b = b^\dagger b$ are non-negative integers. The corresponding eigenkets $|n_a, n_b\rangle$ are energy eigenkets, with energy

$$E_{n_a n_b} = \hbar\omega(1 + n_a + n_b) \tag{4}$$

(a) The ground-state is then

$$|0,0\rangle \qquad E_o = \hbar\omega \tag{5}$$

and the first excited state is doubly degenerate, corresponding to $|1,0\rangle$ and $|0,1\rangle$, with energy $E_{10} = E_{01} = 2\hbar\omega$.

(b) The perturbation to the Hamiltonian can be written

$$V = \delta m\omega^2 xy = \delta\frac{\hbar\omega}{2}(a + a^\dagger)(b + b^\dagger) \tag{6}$$

We apply non-degenerate perturbation theory to the ground-state, to obtain

$$\begin{aligned} |0,0\rangle' &= |0,0\rangle + \frac{1}{E_o - H_o}\hat{V}|0,0\rangle + O(\delta^2) \\ &= |0,0\rangle - \delta|1,1\rangle \end{aligned} \tag{7}$$

with energy

$$\begin{aligned} E'_{0,0} &= E_{oo} + \overbrace{\langle 0,0|V|0,0\rangle}^{0} + \frac{|\langle 1,1|V|0,0\rangle|^2}{E_{00} - E_{11}} + O(\delta^3) \\ &= E_{00} - \delta^2\frac{\hbar\omega}{8} + O(\delta^3) \end{aligned} \tag{8}$$

Next we apply degenerate perturbation theory to the degenerate excited states $|1,0\rangle$ and $|0,1\rangle$. In this manifold of states, the perturbation has matrix elements

$$V_{ab} = \frac{\delta\hbar\omega}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

To zeroth order, the new energy eigenkets are the eigenkets of V_{ab}, given by

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle) \tag{10}$$

with energy eigenvalues

$$E_{\pm} = 2\hbar\omega \pm \frac{\delta\hbar\omega}{2} \tag{11}$$

(c) To solve this problem exactly, we merely have to rotate our spatial axes through 45^o, writing

$$p_{\pm} = \frac{p_x \pm p_y}{\sqrt{2}}, \qquad X_{\pm} = \frac{x \pm y}{\sqrt{2}}. \tag{12}$$

Then,

$$H = \frac{1}{2m}(P_+^2 + P_-^2) + \frac{m\omega_+^2}{2}X_+^2 + \frac{m\omega_-^2}{2}X_-^2 \tag{13}$$

where $\omega_{\pm} = \omega(1 \pm \delta)^{1/2}$, so that if we write

$$\begin{aligned} \alpha &= \sqrt{\frac{m\omega_+}{2\hbar}}\left(X_+ + i\frac{p_+}{m\omega_+}\right) \\ \beta &= \sqrt{\frac{m\omega_-}{2\hbar}}\left(X_- + i\frac{p_-}{m\omega_-}\right) \end{aligned} \tag{14}$$

we may cast the Hamiltonian in the form

$$H = \hbar\omega_+(n_+ + \frac{1}{2}) + \hbar\omega_-(n_- + \frac{1}{2}) \tag{15}$$

where $n_+ = \alpha^\dagger\alpha$ and $n_- = \beta^\dagger\beta$. The ground-state energy is now

$$E_{00} = \frac{1}{2}(\hbar\omega_+ + \hbar\omega_-) = \hbar\omega(1 - \frac{\delta^2}{8}) + O(\delta^3) \tag{16}$$

confirming (8). The excited states have one α or β quanta, and have energies

$$E_{\pm} = E_o + \hbar\omega_{\pm} = \frac{1}{2}(\hbar\omega_+ + \hbar\omega_-) + \hbar\omega_{\pm} \tag{17}$$

To leading order in δ, this gives

$$E_{\pm} = (2 \pm \frac{\delta}{2})\hbar\omega \tag{18}$$

confirming (11).

Let us now construct the new ground-state in terms of the unperturbed ground-state. We first note that

$$\alpha|0,0\rangle' = \beta|0,0\rangle = 0 \tag{19}$$

With a bit of work, by rewriting (14) in terms of the original creation and annihilation operators, you may confirm that

$$\alpha = u_+\left(\frac{a+b}{\sqrt{2}}\right) + v_+\left(\frac{a^\dagger + b^\dagger}{\sqrt{2}}\right)$$

$$\beta = u_- \left(\frac{a-b}{\sqrt{2}} \right) + v_- \left(\frac{a^\dagger - b^\dagger}{\sqrt{2}} \right) \tag{20}$$

where

$$u_\pm = \frac{1}{2}\left[\sqrt{\frac{\omega_\pm}{\omega}} + \sqrt{\frac{\omega}{\omega_\pm}} \right], \qquad v_\pm = \frac{1}{2}\left[\sqrt{\frac{\omega_\pm}{\omega}} - \sqrt{\frac{\omega}{\omega_\pm}} \right]. \tag{21}$$

Note that $u_\pm^2 - v_\pm^2 = 1$. Now to construct the new ground-state, consider the state $|\psi\rangle = e^{Aa^\dagger a^\dagger}|0\rangle$. Now since $[a, e^{Aa^\dagger a^\dagger}] = 2Aa^\dagger e^{Aa^\dagger a^\dagger}$, it follows that

$$(ua + va^\dagger)e^{Aa^\dagger a^\dagger}|0\rangle = (2Au + v)a^\dagger e^{Aa^\dagger a^\dagger}|0\rangle \tag{22}$$

so that

$$(ua + va^\dagger)e^{-\frac{v}{2u}a^\dagger a^\dagger}|0\rangle = 0 \tag{23}$$

Using this result, we can satisfy conditions (19) by writing

$$|0,0\rangle = \exp\left[-\frac{v_+}{2u_+}\left(\frac{a^\dagger + b^\dagger}{\sqrt{2}} \right)^2 - \frac{v_-}{2u_-}\left(\frac{a^\dagger - b^\dagger}{\sqrt{2}} \right)^2 \right]|0,0\rangle \tag{24}$$

Expanding this expression to leading order, gives

$$|0,0\rangle = |0,0\rangle - \left[\frac{v_+}{2u_+}\left(\frac{a^\dagger + b^\dagger}{\sqrt{2}} \right)^2 + \frac{v_-}{2u_-}\left(\frac{a^\dagger - b^\dagger}{\sqrt{2}} \right)^2 \right]|0,0\rangle \tag{25}$$

Now to leading order, $u_\pm = 1 + O(\delta^2)$, $v_\pm = \pm\delta/2 + O(\delta^3)$. Substituting into the above expression, we obtain

$$|0,0\rangle = |0,0\rangle - \delta a^\dagger b^\dagger |0,0\rangle = |0,0\rangle - \delta|1,1\rangle + O(\delta^2) \tag{26}$$

which confirms the perturbative result (7). Finally, the eigenkets of the excited states are obtained by creating quanta:

$$\begin{aligned} \alpha^\dagger|0,0\rangle' &= \frac{1}{\sqrt{2}}(a^\dagger + b^\dagger)|0,0\rangle + O(\delta^2) \\ \beta^\dagger|0,0\rangle' &= \frac{1}{\sqrt{2}}(a^\dagger - b^\dagger)|0,0\rangle + O(\delta^2) \end{aligned} \tag{27}$$

confirming (10).

2. (a) In the degenerate manifold of p-states,

$$V = \lambda(x^2 - y^2) \tag{28}$$

has the matrix elements

$$\langle m|\lambda(x^2 - y^2)|m'\rangle = c\langle m|J_x^2 - J_y^2|m'\rangle = \frac{c}{2}\langle m|J_+^2 - J_-^2|m'\rangle \tag{29}$$

where $|m\rangle \equiv |l = 1, m\rangle$, and we have used the Wigner Eckart theorem to relate the matrix elements to those of the angular momentum operator. The quantity "c" is a constant. From the above, we see that the two only non-vanishing matrix elements are

$$\langle +1|V| - 1\rangle = \langle -1|V|1\rangle = \lambda V_o \tag{30}$$

where V_o is real, due to time-reversal invariance. The zeroth order energy eigenstates are the eigenstates $|\xi\rangle$ of

$$\langle a|V|b\rangle = \begin{bmatrix} 0 & 0 & \lambda V_o \\ 0 & 0 & 0 \\ \lambda V_o & 0 & 0 \end{bmatrix} \tag{31}$$

with energy $E = E_o + \langle\xi|V|\xi\rangle + O(\lambda^3)$ which gives

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|+1\rangle + |-1\rangle), \qquad && E = E_0 + \lambda V_0 + O(\lambda^2) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|+1\rangle - |-1\rangle), \qquad && E = E_0 - \lambda V_0 + O(\lambda^2) \\ |0\rangle & \qquad E = E_0. \end{aligned} \tag{32}$$

showing that the degeneracy of the p-states is completely removed.

(b) Under the time-reversal transformation,

$$\Theta|l, m_l\rangle = (i)^{2m_l}|l, -m_l\rangle = (-1)^{m_l}|l, -m_l\rangle \tag{33}$$

so that

$$\begin{aligned} \Theta|\pm 1\rangle &= -|\mp 1\rangle \\ \Theta|0\rangle &= |0\rangle \end{aligned} \tag{34}$$

which means that

$$\begin{aligned} \Theta\left(\frac{|+1\rangle \pm |-1\rangle}{\sqrt{2}}\right) &= \mp\left(\frac{|+1\rangle \pm |-1\rangle}{\sqrt{2}}\right), \\ \Theta|0\rangle &= |0\rangle \end{aligned} \tag{35}$$

proving that the energy eigenstates are eigenkets of the time-reversal operator.

3. Sakurai 5.17

(a) Let us write the Hamiltonian as $H = H_o + CV$, where

$$\begin{aligned} H_o &= AL^2 + BL_z \\ V &= L_y = \left(\frac{L_+ - L_-}{2i}\right) \end{aligned} \tag{36}$$

The eigenkets of H_o are simultaneous eigenkets of L^2 and L_z, $|lm\rangle$, with energy $E_{lm} = A\hbar^2 l(l+1) + B\hbar m$. We can apply non-degenerate perturbation theory. Now since V is off-diagonal in the $|lm\rangle$ basis, the first order energy shift vanishes, and we are left with

$$\begin{aligned} E'_{lm} &= E^{(o)}_{lm} + C^2 \sum_{m'} \frac{\langle\langle lm'|\left(\frac{L_+ - L_-}{2i}\right)|lm\rangle|^2}{E^{(o)}_{lm} - E^{(o)}_{lm'}} + O(C^3) \\ &= E^{(o)}_{lm} + (C\hbar)^2 \frac{(l-m)(l+m+1)}{-4B\hbar} + (C\hbar)^2 \frac{(l+m)(l-m+1)}{4B\hbar} + O(C^3) \\ &= E^{(o)}_{lm} + \hbar m \frac{C^2}{2B}. + O(C^3) \end{aligned} \tag{37}$$