GRADUATE QUANTUM MECHANICS: 502 Spring 2002

Solution to Assignment 6

1. (a) To control the operator $G = 1/(E - H_o)$ we must introduce a convergence factor

$$G^{\pm} = \frac{1}{E - H_o \pm i\delta} \tag{1}$$

Let us examine the matrix elements of

$$G^{(+)}(x - x') = \frac{\hbar^2}{2m}\langle x|\frac{1}{E - H_o + i\delta}|x'\rangle. \tag{2}$$

Expanding in a momentum space basis, we have

$$\begin{aligned} G^{(+)}(x - x') &= \frac{\hbar^2}{2m}\int dk \langle x|k\rangle\langle k|\frac{1}{E - H_o + i\delta}|k\rangle\langle k|x'\rangle \\ &= \int \frac{dk}{2\pi} e^{ik(x-x')}\frac{1}{k_o^2 - k^2 + i\delta} \end{aligned} \tag{3}$$

where $\hbar^2 k_0^2/2m$. The poles of the integral occur at $k = \pm(k_0 + i\delta)$. Carrying out the integral by contour integration, we must be careful to close the contour in the upper half complex plane for $x > x'$ and the lower-half plane for $x < x'$. This then gives

$$G^{+}(x - x') = \frac{-i}{2k_0} e^{ik_0|x-x'|} \tag{4}$$

This describes a wave that is moving *outwards* from the point of origin x'. Had we chosen $-i\delta$, rather than $+i\delta$ in the denominator of Greens function we would have found that the Green-function described an *incoming wave*. By using the $+i\delta$ scheme, we obtain the correct form for the scattered wave. The Lippmann Schwinger equation now becomes

$$\psi^{(+)}(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{2m}{\hbar^2}\int dx' G^{(+)}(x - x')V(x')\psi^{(+)}(x') \tag{5}$$

For $x < 0$, this equation describes an incoming and reflected wave. For $x > a$, $\psi^{(+)}$ describes an incoming and transmitted wave.

(b) For the special case of an attractive delta-function potential

$$V(x) = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x), \qquad (\gamma > 0), \tag{6}$$

the integral equation becomes

$$\psi^{(+)}(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k_0} G^{(+)}(x)\psi^{(+)}(0). \tag{7}$$

Setting $x = 0$ in this expression , we obtain

$$\psi^{(+)}(0) = \frac{1}{\sqrt{2\pi}}\frac{2k_0}{2k_0 - i\gamma} \tag{8}$$

so that

$$\psi^{(+)}(x) = \frac{1}{\sqrt{2\pi}}\left[e^{ik_0 x} + t(k_0)e^{ik_0|x|}\right] \tag{9}$$

where

$$t(k_0) = \frac{i\gamma}{2k_0 - i\gamma} \tag{10}$$

is the t-matrix scattering amplitude.

To see how to convert the t-matrix into the S-matrix, let us note that a plane wave in one dimension can be decomposed into a "symmetric" and "antisymmetric" scattering channel, as follows:

$$\begin{aligned} e^{ikx} &= \frac{1}{2}[\overbrace{e^{ik|x|}}^{\text{outgoing}} + \underbrace{e^{-ik|x|}}_{\text{incoming}}] && \text{(symmetric)} \\ &+ \frac{1}{2}sgn(x)[e^{ik|x|} - e^{-ik|x|}] && \text{(antisymmetric)} \end{aligned} \tag{11}$$

When scattering takes place, the out-going waves will pick up a phase shift, so asymptotically far from the scattering center, we expect

$$\psi^+(x) = \frac{1}{2}[\overbrace{e^{2i\delta_+}}^{S^+(k)=e^{2i\delta^+}} e^{ik|x|} + e^{-ik|x|}] + \frac{1}{2}sgn(x)[\overbrace{e^{2i\delta_-}}^{S^-(k)=e^{2i\delta^-}} e^{ik|x|} + -e^{-ik|x|}] \tag{12}$$

If we expand this expression out, we find that

$$\psi^{(+)}(x) = e^{ikx} + t^{(+)}(k)e^{ik|x|} + t^{(-)}(k)e^{ik|x|}\text{sign}(x) \tag{13}$$

where

$$t^{\pm}(k) = \frac{1}{2}(S^{\pm}(k) - 1) = \frac{1}{2}[e^{2i\delta^{\pm}} - 1] \tag{14}$$

For the problem considered here, the scattering potential is symmetric under parity inversion, so that $\delta_- = 0$ is zero and there is no scattering in the antisymmetric channel. ($t^-(k) = 0$). The S-matrix in the symmetric scattering channel is then

$$S^{(+)}(k) = 2t^{(+)}(k) + 1 = \frac{2k_0 + i\gamma}{2k_0 - i\gamma} \tag{15}$$

(c) Notice that the t-matrix and S-matrix has a singular pole at $k_0 = i\gamma/2$. The corresponding energy is then

$$E = -E_B = \frac{\hbar^2 k_0^2}{2m} = -\frac{\hbar^2\gamma^2}{8m}. \tag{16}$$

which corresponds to the single binding energy of a delta-function attractive well.

2. (Sakurai Chapter 7, Problem 3) In the potential

$$V = \begin{cases} 0, & (r > R) \\ V_o, & (r < R) \end{cases} \tag{17}$$

where V_o may be positive or negative, the radial wavefunction in the channel with angular momentum l is given by

$$\psi(r) \propto \begin{cases} j_l(\tilde{k}r), & (r < R) \\ e^{i\delta}(\cos\delta_l j_l(kr) - \sin\delta_l \eta_l(kr)), & (r > R) \end{cases} \tag{18}$$

where

$$\tilde{k} = \sqrt{k^2 - \frac{2mV_o}{\hbar^2}} \tag{19}$$

By matching the logarithmic derivative of the wavefunction at $r = R$, we obtain

$$\tilde{k}R\frac{j_l(\tilde{k}R)}{j_l(\tilde{k}R)} = kR\frac{\cos\delta_l j_l'(kR) - \sin\delta_l \eta_l'(kR)}{\cos\delta_l j_l(kR) - \sin\delta_l \eta_l(kR)} \tag{20}$$

Solving this equation, we obtain

$$\begin{aligned}\tan\delta_l &= \frac{j_l(kR)}{\eta_l(kR)}\left[\frac{xj_l'/j_l - \tilde{x}\tilde{j}_l'/\tilde{j}_l}{x\eta_l'/\eta_l - \tilde{x}\tilde{j}_l'/\tilde{j}_l}\right] \\ &= \frac{j_l(kR)}{\eta_l(kR)}\left[\frac{\frac{d\ln j_l}{d\ln x} - \frac{d\ln\tilde{j}_l}{d\ln\tilde{x}}}{\frac{d\ln\eta_l}{d\ln x} - \frac{d\ln\tilde{j}_l}{d\ln\tilde{x}}}\right]\end{aligned} \tag{21}$$

where I have used the shorthand notation $x = kR$, $\tilde{x} = \tilde{k}R$, $j_l \equiv j_l(x)$, $\tilde{j}_l \equiv j_l(\tilde{x})$ and so on. Now at low energies ($k \to 0$), the quantity in square brackets goes to a constant. Since

$$j_l(x) \sim \frac{x^l}{(2l+1)!!}, \qquad \eta_l(x) \sim -\frac{(2l-1)!!}{x^{l+1}}, \tag{22}$$

the prefactor becomes

$$\frac{j_l}{\eta_l} \sim -\frac{(kR)^{2l+1}}{(2l+1)[(2l-1)!!]^2} \tag{23}$$

so that at very low energies we may restrict our attention to the s-channel (l=0).

(a) Now for small values of x and $\tilde{x}$,

$$\begin{aligned}j_o &\approx 1 - \frac{x^2}{3!} \Rightarrow d\ln j_o/d\ln x = -\frac{x^2}{3} \\ \eta_o &\approx -\frac{1}{x}\left(1 - \frac{x^2}{2}\right) \Rightarrow d\ln\eta_0/d\ln x = -1 - x^2\end{aligned} \tag{24}$$

so that

$$\tan\delta_o = -x\left[\frac{(\tilde{x}^2 - x^2)/3}{-1}\right] = x(\tilde{x}^2 - x^2)/3 + O(x^3) = -\frac{kR}{3}\left(\frac{2mV_oR^2}{\hbar^2}\right) + O(x^3) \tag{25}$$

From this result, we may compute the scattering cross-section, which is dominated by the s-wave scattering amplitude and given by

$$\sigma_{tot} \approx \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\tan^2\delta_0 = \frac{16\pi}{9}\left(\frac{mV_oR^3}{\hbar^2}\right)^2 \tag{26}$$

(b) To calculate the angular dependence of the scattering cross-section at higher energies, we need to include the $l = 1$ p-wave scattering. The differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{27}$$

Expanding the scattering amplitude in partial waves, we have

$$\begin{aligned} f(\theta) &= \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \\ &= \frac{1}{k}\left[e^{i\delta_o}\sin\delta_o + 3e^{i\delta_1}\sin\delta_1\cos\theta\right] \end{aligned} \tag{28}$$

where it has been truncated beyond $l = 1$. Thus to leading order, the low-energy differential scattering cross-section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{k^2}\left[\sin^2\delta_o + 6\sin\delta_1\sin\delta_0\cos(\delta_1-\delta_0)\cos\theta\right] \\ &= \frac{1}{k^2}(\delta_o)^2\left[1 + 6(\delta_0/\delta_1)\cos\theta\right] \end{aligned} \tag{29}$$

To calculate the p-wave phase shift, we use equation (21). At low energies, we have

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \approx \frac{x}{3}\left[1 - \frac{x^2}{10}\right] \Rightarrow \frac{d\ln j_1}{d\ln x} = 1 - \frac{x^2}{5} + O(x^4) \tag{30}$$

Similarly,

$$\eta_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x} \approx -\frac{1}{x^2} - \frac{1}{2} \Rightarrow \frac{d\ln\eta_1}{d\ln x} = -2 + x^2 + O(x^4) \tag{31}$$

so that the p-wave scattering phase shift is

$$\begin{aligned} \tan\delta_1 &= \frac{j_1}{\eta_1}\left[\frac{\frac{d\ln j_1}{d\ln x} - \frac{d\ln\tilde{j}_1}{d\ln\tilde{x}}}{\frac{d\ln\eta_1}{d\ln x} - \frac{d\ln\tilde{j}_1}{d\ln\tilde{x}}}\right] = \frac{x^2}{45}(\tilde{x}^2 - x^2) \\ &= \frac{(kR)^2}{45}\left(-\frac{2mV_oR^2}{\hbar^2}\right) = \frac{kR}{15}\tan\delta_0 \end{aligned} \tag{32}$$

So we see that $\delta_1 = \frac{kR}{15}\delta_0$ at low energies. Inserting these results into (29) , we obtain finally that

$$\frac{d\sigma}{d\Omega} \approx \frac{\delta_o^2}{k^2}\left[1 + \frac{2}{5}kR\cos\theta\right] \tag{33}$$

or

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta \tag{34}$$

where

$$\begin{aligned} A &= \frac{\sigma_{tot}}{4\pi} = \frac{4}{9}\left(\frac{mV_oR^3}{\hbar^2}\right)^2 \\ \frac{B}{A} &= \frac{2}{5}kR \end{aligned} \tag{35}$$

These results represent the leading quadratic $O(V)^2$ contribution to the scattering cross-section, they will be completely captured by the low energy limit of the Born Scattering cross-section.