GRADUATE QUANTUM MECHANICS: 502 Spring 2002

Solutions to Assignment 2.

1. To find the Clebsch Gordan coefficients, we start with the state formed by combining parallel spins, then we apply the lowering operator successively. In the following solutions, the Clebsh-Gordan coefficients can be read off from the corresponding expansion of the angular momentum states.

   When we make the combination $\frac{1}{2} \times \frac{3}{2}$ we generate a $j = 2$ and a $j = 1$ state. The total number of states is $2 \times 4 = 8 = 5 + 3$, so the $j = 1$ and $j = 2$ state exhaust the total number of states.

$$\frac{1}{2} \otimes \frac{3}{2} = 2 \oplus 1 \tag{1}$$

   The state with maximum m_z corresponding formed from $j = \frac{1}{2}$ and $j = \frac{3}{2}$ is

$$|\mathbf{2}, 2\rangle = |\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle \tag{2}$$

   If we apply the lowering operator to this state we obtain

$$\begin{aligned} J_-|\mathbf{2}, 2\rangle &= 2|\mathbf{2}, 1\rangle && (3)\\ J_-|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle &= |\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle(J_-|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle + (J_-|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle && (4)\\ &= |\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle + \sqrt{3}|\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle, && (5) \end{aligned}$$

   so that

$$|\mathbf{2}, 1\rangle = \frac{1}{2}|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle + \frac{\sqrt{3}}{2}|\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle \tag{6}$$

   Repeating this process, we obtain

$$\begin{aligned} |\mathbf{2}, 2\rangle &= |\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle && (7)\\ |\mathbf{2}, 1\rangle &= \frac{1}{2}|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle + \frac{\sqrt{3}}{2}|\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle && (8)\\ |\mathbf{2}, 0\rangle &= \frac{1}{\sqrt{2}}\left[|\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle + |\mathbf{\tfrac{3}{2}}, -\tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle\right] && (9)\\ |\mathbf{2}, -1\rangle &= \frac{1}{2}\left[\sqrt{3}|\mathbf{\tfrac{3}{2}}, -\tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle + |\mathbf{\tfrac{3}{2}}, -\tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle\right] && (10)\\ |\mathbf{2}, -2\rangle &= |\mathbf{\tfrac{3}{2}}, -\tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle && (11) \end{aligned}$$

   Now, to obtain the spin-1 states, we write

$$|\mathbf{1}, 1\rangle = \alpha|\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle + \beta|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle \tag{12}$$

   In order that this state be orthogonal to $|\mathbf{2}, 1\rangle$, we choose

$$|\mathbf{1}, 1\rangle = \frac{1}{2}\left[\sqrt{3}|\mathbf{\tfrac{3}{2}}, \tfrac{3}{2}\rangle|\mathbf{\tfrac{1}{2}}, -\tfrac{1}{2}\rangle - |\mathbf{\tfrac{3}{2}}, \tfrac{1}{2}\rangle|\mathbf{\tfrac{1}{2}}, \tfrac{1}{2}\rangle\right] \tag{13}$$

Now, successive application of the lowering operator yields

$$|\mathbf{1},0\rangle = \frac{1}{\sqrt{2}}|\frac{\mathbf{3}}{\mathbf{2}},\frac{1}{2}\rangle|\frac{\mathbf{1}}{\mathbf{2}},-\frac{1}{2}\rangle - |\frac{\mathbf{3}}{\mathbf{2}},-\frac{1}{2}\rangle|\frac{\mathbf{1}}{\mathbf{2}},\frac{1}{2}\rangle \tag{14}$$

$$|\mathbf{1},-1\rangle = \frac{-\sqrt{3}}{2}|\frac{\mathbf{3}}{\mathbf{2}},-\frac{3}{2}\rangle|\frac{\mathbf{1}}{\mathbf{2}},\frac{1}{2}\rangle + \frac{1}{2}|\frac{\mathbf{3}}{\mathbf{2}},-\frac{1}{2}\rangle|\frac{\mathbf{1}}{\mathbf{2}},-\frac{1}{2}\rangle \tag{15}$$

2. Sakurai, problem 3.18. The probability that the rotated state $D(R)|l=2,m=0\rangle$ has $J_z = m'$ is given by

$$p(m') = |\langle l=2,m'|D(R)|l=2,m=0\rangle|^2 = |D^{(2)}_{m'0}(R)|^2 \tag{16}$$

But since $D^{(2)}_{m'0} = \sqrt{\frac{4\pi}{5}}Y^*_{2m'}(R)$, we obtain

$$p(m') = \frac{4\pi}{5}|Y^*_{2m'}(\theta,\phi)|^2 \tag{17}$$

Writing this out explicitly, we obtain

$$p(\pm 2) = \frac{3}{8}\sin^4\theta, \tag{18}$$

$$p(\pm 1) = \frac{3}{2}\sin^2\theta\cos^2\theta, \tag{19}$$

$$p(0) = \frac{1}{4}(3\cos^2\theta - 1). \tag{20}$$

3. Sakurai, problem 3.24. The singlet wavefunction of the two spins can be written

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\hat{n},+\rangle_1|\hat{n},-\rangle_2 - |\hat{n},-\rangle_1|\hat{n},+\rangle_2\right) \tag{21}$$

where $\hat{n}$ is an arbitrary observation axis.

(a) When observer B makes no measurement, the state remains in a linear superposition of the two states $|z,\pm\rangle_1|z,\mp\rangle_2$ with an amplitude of magnitude $\frac{1}{\sqrt{2}}$ to be either state. The probability for A to obtain $s_{1z} = \hbar/2$ is thus precisely $\frac{1}{2}$. This holds for any quantization axis, so the probability for measuring $s_{1x} = \hbar/2$ is also precisely $\frac{1}{2}$.

(b) If B measures the spin of the particle to be $s_{2z} = \hbar/2$, this projects the system into the state $|z,-\rangle_1|z,+\rangle_2$, so that A must measure $s_{1z} = -\hbar/2$. If A measures s_{1x}, then since

$$|z,-\rangle_1|z,+\rangle_2 = \frac{1}{\sqrt{2}}\left(|x,+\rangle_1|z,+\rangle_2 + |x,-\rangle_1|z,+\rangle_2\right) \tag{22}$$

there is an equal chance that A will measure $s_{1x} = \pm\hbar/2$.

4. (a) Now using the Wigner Eckart theorem, we can write

$$Q_a b = e\langle jm|x_a x_b - \frac{1}{3}r^2|jm\rangle = \frac{\lambda}{\hbar^2}\langle jm|J_a J_b - \frac{1}{3}J^2|jm\rangle \tag{23}$$

Now if we evaluate $\langle jj|Q_{33}|jj\rangle$, we obtain

$$\langle jj|Q_{33}|jj\rangle = \frac{e}{3}\langle jj|z^2 - r^2|jj\rangle = 3Q \tag{24}$$

on the other hand, from the r.h.s. of the first expression, we obtain

$$\lambda\langle jj|J_z^2 - \frac{1}{3}J^2|jj\rangle = \lambda\frac{j(2j-1)}{3} \tag{25}$$

Comparing these two results, we obtain

$$\lambda = \frac{Q}{2(2j-1)} \tag{26}$$

(b) From the previous part of the question, we have

$$\begin{aligned} e\langle jm|x^2 - y^2|jm'\rangle &= \frac{Q}{\hbar^2 j(2j-1)}\langle jm|J_x^2 - J_y^2|jm'\rangle \\ &= \frac{Q}{\hbar^2 2j(2j-1)}\langle jm|J_+^2 - J_-^2|jm'\rangle \end{aligned} \tag{27}$$

We can write this in the form

$$e\langle jm|x^2 - y^2|jm'\rangle = \frac{Q}{\hbar^2 2j(2j-1)}\left[\delta_{m,m'-2}\langle jm|J_+^2|jm+2\rangle + \delta_{m,m'+2}\langle jm|J_-^2|jm-2\rangle\right] \tag{28}$$

For the specific case of $j = 1$, we have

$$\langle 11|J_+2|1,-1\rangle = \langle 1-1|J_+2|1,+1\rangle = 2\hbar^2 \tag{29}$$

so that

$$e\langle 1m|x^2 - y^2|1m'\rangle = Q\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{30}$$