GRADUATE QUANTUM MECHANICS: 502 Spring 2002

Take-home final. Please leave your finished answers in my mail box by **4pm, Monday May 13th**. Please try to answer all questions. Feel free to refer to text books, but please- no consultation!

1. Suppose the electron were a spin 3/2 particle obeying Fermi-Dirac statistics.

   (a) Write the configuration of a hypothetical Ne (Z=10) atom made up of such "electrons". (That is, the $S = 3/2$ analog of $(1s)^2(2s)^2(2p)^6$). Show that the configuration is highly degenerate.

   (b) If exchange and spin-orbit splitting are taken into account, what ground-state configuration is selected? Please write your answer in spectroscopic notation? (${}^{2S+1}L_J$, where S, L and J stand for the total spin, the total orbital angular momentum, and the total angular momentum respectively.)

2. Consider a particle in one dimension moving under the influence of some time-independent potential. The energy levels and the corresponding eigenfunctions for this problem are assumed to be known. We now subject the particle to a pulse travelling at speed c, represented by a time-dependent potential,

$$V(t) = A\delta(x - ct)$$

   (a) Suppose at $t = -\infty$ the particle is known to be in the ground-state whose energy eigenfunction is $\langle x|i\rangle = u_i(x)$. Obtain the probability for finding the system in some excited state with energy eigenfunction $u_f(x) = \langle x|f\rangle$ at $t = +\infty$.

   (b) Interpret your result in (a) physically by regarding the delta function pulse as a superposition of harmonic perturbations; recall

$$\delta(x - ct) = \frac{1}{2\pi c}\int_{-\infty}^{\infty} d\omega e^{i\omega[(x/c)-t]}.$$

   Emphasize the role played by energy conservation, which holds even quantum mechanically as long as the perturbation has been on for a very long time.

3. A rotator whose orientation is specified by the angular co-ordinates θ and ϕ rotates in a potential that favors orientation in the $y - z$ plane. The Hamiltonian is given by

$$H = A\mathbf{L}^2 + B\hbar^2\cos 2\phi \tag{1}$$

   with $A >> B$.

   (a) Calculate the s (l=0), p(l=1) and d(l=2) energy levels of this system in first order perturbation theory.

   (b) Write down the leading order expressions for the corresponding energy eigenkets.

4. (a) Verify that, outside the range of a short-range potential, the wave-function

$$u(r,\theta) = \frac{1}{r}\left(1 + \frac{i}{kr}\right)e^{ikr}\cos\theta \tag{2}$$

   represents an outgoing p-wave.

   (b) A beam of particles represented by the plane wave e^{ikz} is scattered by an impenetrable sphere of radius a, where $ka << 1$. Considering only s and p components, and working to order $(ka)^2$, obtain an expression for the differential scattering cross-section for scattering at an angle θ.

5. The Dirac equation for a relativistic fermion is taken to be

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-ic\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right]\psi \tag{3}$$

where

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{bmatrix}.$$

are the 4×4 Dirac matrices and ψ is a four-component spinor.

(a) Derive the properly normalized four component wave-function for a positive energy spin-up particle travelling in the z-direction with momentum p. You may use the notation $E(p) = \sqrt{m^2c^4 + p^2c^2}$. Please normalize your wavefunction to one, assuming that the wave is travelling in a box of side-length 1.

(b) Derive the normalized four-component wave-function for a negative energy spin-up particle travelling in the z-direction with momentum p.

(c) Suppose that a particle has wavefunction at time $t = 0$

$$\psi(\vec{x}, 0) = \frac{1}{L^{3/2}}\begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix} e^{i\vec{p}\cdot\vec{x}/\hbar}$$

where $\vec{p} = (0, 0, p)$ is along the z-axis. Calculate the form of the wavefunction at later times <u>and</u> give the probabilities $p^{\pm}$ for the particle to be in a positive, or negative energy state.