1. Short questions.

(a) If $|i\rangle$ and $|j\rangle$ are eigenkets of Hermitian operator $A$. Under what conditions is $|i\rangle + |j\rangle$ an eigenket of $A$?

(b) A beam of intensity $I$ carrying spin 1/2 atoms polarized in the $+z$ direction passes through two Stern Gerlach type measurements. The first measurement only accepts atoms with $S_n = \hbar/2$, where $S_n = \mathbf{S} \cdot \hat{n}$ is the spin component along an axis $\hat{n}$ at an angle $\beta$ to the $z$-axis. The second measurement only accepts “down-spin” atoms with $S_z = -\hbar/2$. What is the final beam intensity?

(c) If $f(A)$ is a function of an operator $A$ with eigenkets $|a\rangle$ where $A|a'\rangle = a'|a\rangle$, write down an expression for the matrix elements $\langle b'\mid f(A)\mid b''\rangle$ of $f(A)$ in a new basis where the matrix elements relating the $|a\rangle$ and $|b\rangle$ basis are known.

(d) The states $|1\rangle$ and $|2\rangle$ are energy eigenstates with energies $E_1$ and $E_2$. The operator $A$ has eigenkets $|+\rangle$ and $|-\rangle$ given by $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$, where $A$ has values $a_+$ and $a_-$ respectively. Calculate the time dependence of the expectation value $\langle A(t) \rangle$.

(e) An electron moves in one dimension a potential $V(x) = -\lambda x$. Give an approximate mathematical form for the wavefunction and sketch it, taking care to show how the amplitude varies with position. Is the spectrum bounded or unbounded?

2. A particle moves in a spherically symmetric potential, and has wavefunction

$$\psi(x, y, z) = \langle x, y, z \mid \psi \rangle = f(r)(x + y + 3z). \quad (1)$$

(a) Is this state an eigenstate of $L^2$? Explain your answer.

(b) If the component of angular momentum in the $z$ direction is measured, what values can be obtained, and what will their probability be?

(c) If $|\psi\rangle$ is an energy eigenstate with energy $E_0$, use the wavefunction to derive the corresponding potential $V(r)$.

3. This is a question about positronium, a bound-state of an electron and a positron. Since positrons and electrons have the same mass, we have to take into account the motion of both particles.

(a) For two particles of mass $m_1$ and $m_2$ show that the total momentum $P = p_1 + p_2$ and the center of mass position

$$X = \frac{(m_1 x_1 + m_2 x_2)}{M}, \quad (2)$$

where $M = m_1 + m_2$, are canonically conjugate.

(b) Show that the relative position $x = x_2 - x_1$ and relative momentum

$$p = \mu \left( \frac{1}{m_1} p_2 - \frac{1}{m_2} p_1 \right), \quad (3)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass, commute with $P$ and $X$ and are also canonically conjugate.

(c) Show that the Hamiltonian for the system

$$H = \frac{(p_1)^2}{2m_1} + \frac{(p_2)^2}{2m_2} + V(|x_2 - x_1|) \quad (4)$$

decouples into two terms $H = H_{CM} + H_{internal}$ where

$$H_{CM} = \frac{1}{2M} P^2, \quad H_{internal} = \frac{1}{2\mu} p^2 + V(|x|) \quad (5)$$
(d) Use the equations of motion for $\mathbf{P}$ and $\mathbf{p}$ to show that (i) the center of mass momentum is conserved and (ii) the internal motion is equivalent to a single single particle of mass $\mu$ moving about a fixed potential $V(r)$, where $r = |\mathbf{x}|$.

(e) Working by analogy with the hydrogen atom, give expressions for (a) the bound-state energies and (b) the “Bohr” radius associated with the ground-state wavefunction of positronium. What are the approximate numerical sizes of these quantities?

(f) When a positron encounters an electron, they annihilate into two photons. Which angular momentum states of positronium will be the most unstable? Explain your answer carefully.