Assignment 8.

Read Shankar Chapter 13.

1. A spin $\frac{1}{2}$ is initially in the state $|z+,\rangle$. The spin is rotated (i) through angle $\pi/2$ about the $y$ axis, then (ii) through angle $\pi/2$ about the $z$ axis, and finally (iii) through angle $\pi/2$ about the $x$ axis. What is state after each of these three stages? Does the spin return to precisely the initial state?

2. An ensemble of electrons have spins which point 50% in the $+y$ direction and 50% in the $+z$ direction.
   (a) what is the spin density matrix for this ensemble?
   (b) What are the expectation values for the three components of the spin $\langle S_x, S_y, S_z \rangle$?
   (c) If a magnetic field of strength $B$ is applied in the $+z$ direction, what is the spin density matrix after time $t$?

3. In the state $|jm\rangle, J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle$ and $J_z|jm\rangle = \hbar m|jm\rangle$.
   (a) Show that in this state, $\langle J_x \rangle = \langle J_y \rangle = 0$.
   (b) Show that in this state
      $$\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{1}{2}\hbar^2 [j(j+1) - m^2]$$
   (c) Show that $\Delta J_x$ and $\Delta J_y$ are consistent with Heisenberg’s uncertainty relation for angular momentum.
   (d) Show that the states $|j, \pm j\rangle$ are minimum uncertainty wavepackets.

4. A particle with angular momentum $l = 0$ moves in an attractive central potential field with potential given by
   $$U(r) = \begin{cases} 
   -U_0 & r < R \\
   0 & r > R 
   \end{cases}$$
   where $U_0 > 0$.
   (a) Construct the bound-state wavefunctions.
   (b) By matching the logarithmic derivative of the radial wavefunction at $r = R$, show that the binding energy $B$ of each bound-state (where $B = -E > 0$ is the negative of the energy eigenvalue of the bound-state) must satisfy:
      $$\kappa \cot(\kappa R) = -\sqrt{\frac{2mB}{\hbar^2}}$$
      $$\kappa = \sqrt{\frac{2m(U - B)}{\hbar^2}}$$
   (c) What is the smallest value of $U_0$ (i.e the shallowest potential well) for which a single bound-state will form?