GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Solution to Assignment 4.

1. (a) For a free particle, $H = \frac{p^2}{2m}$. The Heisenberg equations of motion are

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{i\hbar}[x, \frac{p^2}{2m}] = \frac{p}{m} \\ \frac{dp}{dt} &= \frac{1}{i\hbar}[p, \frac{p^2}{2m}] = 0 \end{aligned} \tag{1}$$

From which we deduce that

$$p(t) = p, \qquad x(t) = x + \frac{p}{m}t \tag{2}$$

where $p \equiv p(0)$, $x \equiv x(0)$. It thus follows that

$$[x(t), x(0)] = [x + \frac{p}{m}t, x] = \frac{-i\hbar t}{m} \tag{3}$$

(b) From the above result,

$$\begin{aligned} \langle x^2(t)\rangle &= \langle (x + \frac{p}{m}t)^2\rangle \\ &= \langle x^2\rangle + \langle p^2\rangle\frac{t^2}{m^2} + \frac{t}{m}\langle xp + px\rangle \end{aligned} \tag{4}$$

Now

$$\langle x(t)\rangle^2 = \left(\langle x\rangle + \frac{t}{m}\langle p\rangle\right)^2 \tag{5}$$

so subtracting these two results, we obtain

$$\langle x(t)^2\rangle - \langle x(t)\rangle^2 = \langle \Delta x^2(t)\rangle \quad = \quad \langle \Delta x^2\rangle + \langle p^2\rangle\frac{t^2}{m^2} + \frac{t}{m}\langle\{\Delta x,\ \Delta p\}\rangle \tag{6}$$

Now the uncertainty relation tells us that

$$\begin{aligned} \langle \Delta x^2\rangle\langle \Delta p^2\rangle &\geq \left[\frac{-i}{2}\langle [x,p]\rangle\right]^2 + \left[\frac{1}{2}\langle\{\Delta x, \Delta p\}\rangle\right]^2 \\ &= \frac{\hbar^2}{4} + \left[\frac{1}{2}\langle\{\Delta x, \Delta p\}\rangle\right]^2 \end{aligned} \tag{7}$$

In a minimal uncertainty wavepacket, at $t = 0$ $\Delta x \Delta p = \hbar/2$, so the second term is zero, and we may write

$$\Delta x(t)^2 = \Delta x_o^2 + \langle \Delta p^2\rangle\frac{t^2}{m^2} = \Delta x_o^2 + \frac{\hbar^2 t^2}{4m^2\Delta x_o^2} \tag{8}$$

(c) Since $\Delta x(t) = 10^{-15}m >> \Delta x = 10^{-6}m$, we may estimate

$$\begin{aligned} \langle \Delta x(t)^2 \rangle &= \langle \Delta x^2 \rangle + \frac{\hbar^2 t^2}{4m^2 \langle \Delta x^2 \rangle_0} \\ &\approx \frac{\hbar^2 t^2}{4m^2 \langle \Delta x^2 \rangle_0} \end{aligned} \tag{9}$$

so that

$$t \approx \frac{2\Delta x_f \Delta x_o m}{\hbar} = \frac{2.10^{-15}.10^{-6}.10^{-3}kg}{10^{-34}Js} = 2 \times 10^{10}s \approx 600yrs \tag{10}$$

2. In the $\{|R\rangle,\ |L\rangle\}$ basis, the Hamiltonian takes the form

$$H = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} \tag{11}$$

(a) The eigenstates and eigenkets are, by inspection,

$$|\pm\rangle = \frac{|R\rangle \pm |L\rangle}{\sqrt{2}}, \qquad E_\pm = \pm\Delta \tag{12}$$

(b) The time evolution operator can be written

$$e^{-iHt/\hbar} = e^{-i\omega t}|+\rangle\langle+| + e^{+i\omega t/\hbar}|-\rangle\langle-| \tag{13}$$

where $\omega = \Delta/\hbar$. From this result, we have,

$$\begin{aligned} |\alpha(t)\rangle &= e^{-iHt/\hbar}|\alpha\rangle \\ &= e^{-i\omega t}|+\rangle\langle+|\alpha\rangle + e^{+i\omega t}|-\rangle\langle-|\alpha\rangle \\ &= \left(\frac{\alpha_R + \alpha_L}{\sqrt{2}}\right) e^{-i\omega t}|+\rangle + \left(\frac{\alpha_R - \alpha_L}{\sqrt{2}}\right) e^{i\omega t}|-\rangle \\ &= (\alpha_R \cos\omega t - i\alpha_L \sin\omega t)\,|R\rangle + (\alpha_L \cos\omega t - i\alpha_R \sin\omega t)\,|L\rangle \end{aligned} \tag{14}$$

(c) Setting $\alpha_R = 1$, $\alpha_L = 0$, the probability to be in the left side at time t is given by

$$p_L(t) = |\langle L|\alpha(t)\rangle|^2 = \sin^2(\omega t). \tag{15}$$

(d) The Schrödinger equation becomes

$$i\hbar \frac{d}{dt}\begin{pmatrix} \alpha_R \\ \alpha_L \end{pmatrix} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} \alpha_R \\ \alpha_L \end{pmatrix}, \tag{16}$$

or

$$\dot{\alpha}_R = -i\omega\alpha_L, \qquad \dot{\alpha}_L = -i\omega\alpha_R \tag{17}$$

Substituting the second equation into the first gives

$$\ddot{\alpha}_R + \omega^2 \alpha_R = 0 \tag{18}$$

so that

$$\alpha_R(t) = Ae^{-i\omega t} + Be^{i\omega t} \tag{19}$$

From the boundary conditions, $\alpha_R(0) = \alpha_R$, $\dot{\alpha}_R(0) = -i\omega\alpha_L$, we obtain

$$\begin{aligned} A + B &= \alpha_R \\ -i\omega(A - B) &= \omega\alpha_L \end{aligned} \tag{20}$$

so that $A = \frac{1}{2}(\alpha_R + i\alpha_L)$, $B = \frac{1}{2}(\alpha_R - i\alpha_L)$. Simplifying the expression, we obtain

$$\begin{aligned} \alpha_R(t) &= (\alpha_R \cos\omega t - i\alpha_L \sin\omega t)\,, \\ \alpha_L(t) &= (\alpha_L \cos\omega t - i\alpha_R \sin\omega t)\,, \end{aligned} \tag{21}$$

which recovers the result of (b).

(e) If $H = \Delta|R\rangle\langle L|$, then the Schrödinger equation becomes

$$i\hbar\frac{d}{dt}\begin{pmatrix}\alpha_R \\ \alpha_L\end{pmatrix} = \begin{pmatrix}0 & \Delta \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_R \\ \alpha_L\end{pmatrix}, \tag{22}$$

or

$$\dot{\alpha}_R = -i\omega\alpha_L, \qquad \dot{\alpha}_L = 0, \tag{23}$$

so that $\alpha_L(t) = \alpha$, $\alpha_R(t) = \alpha_R - i\omega t\alpha_L$ and then

$$p_R(t) + p_L(t) = |\alpha_L(t)|^2 + |\alpha_R(t)|^2 = 1 + \alpha_L^2\omega^2 t^2 \neq 1 \tag{24}$$

and the total probability is no longer conserved.

3. In this problem, we need to find the solutions to Schrödingers equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V(x) - E)\psi(x) = 0, \qquad V(x) = \begin{cases} \frac{1}{m}\omega^2 x^2 & (x > 0) \\ \infty & (x < 0) \end{cases} \tag{25}$$

Since the potential is infinite for $x < 0$, $\psi(x) = 0$ for $x < 0$. We can impose this condition using the method of images: solving the problem where $V(x) = \frac{1}{2}m\omega^2 x^2$, and taking only odd-parity harmonic oscillator solutions. Properly normalized, this means we must take

$$\psi(x) = \sqrt{2}\psi_{2n+1}(x) \tag{26}$$

where

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{x}{\Delta x} - \Delta x\frac{d}{dx}\right)^n \frac{1}{(\pi\Delta x^2)^{1/4}} e^{-x^2/2\Delta x^2}, \tag{27}$$

and $\Delta x = \sqrt{\frac{\hbar}{m\omega}}$. The ground-state is then

$$\psi_g(x) = \sqrt{2}\psi_1(x) = \frac{2}{(\pi\Delta x^2)^{1/4}}\left(\frac{x}{\Delta x}e^{-x^2/2\Delta x^2}\right)\theta(x) \tag{28}$$

The corresponding ground-state energy is

$$E = \frac{3}{2}\hbar\omega \tag{29}$$

and the average position is

$$\begin{aligned}
\langle x \rangle &= \int_0^\infty dx |\psi(x)|^2 x \\
&= \left(\frac{4}{\sqrt{\pi}} \Delta x \right) \int_0^\infty u^3 e^{-u^2} du \\
&= \frac{2}{\sqrt{\pi}} \Delta x \int_0^\infty x e^{-x} dx \\
&= \sqrt{\frac{4\hbar}{m\omega\pi}}
\end{aligned} \tag{30}$$