GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Assignment 4. (Due Mon, Oct 22nd)

Read Sakurai p. 100-109.

i. Let $\hat{x}(t)$ be the co-ordinate operator for a free particle in 1D in the Heisenberg picture.

(a) Evaluate $[\hat{x}(t), \hat{x}(0)]$.

(b) A minimal wave packet is located at the origin at time $t = 0$ with average momentum $\langle \hat{p} \rangle = p_o$. At $t = 0$, $\Delta x \Delta p = \hbar/2$. Using the Heisenberg equations of motion, obtain $\langle \Delta x^2(t) \rangle$ as a function of time, assuming $\langle \Delta \hat{x}^2(0) \rangle \equiv \Delta x_o^2$ is given.

(c) A $1g$ particle is located at a certain location with an accuracy equal to the diameter of a proton ($10^{-15} m$). Estimate how long will it take for the uncertainty in position to grow to a micron.

ii. (Variant on Sakurai, problem 9, Ch 2.) An electron in a symmetric double-well potential with two identical minima, can sit either in the left or right potential well. A particle on the right or left-hand side is represented by the eigenket $|R\rangle$ and $|L\rangle$ respectively. The most general state vector can then be written as $|\alpha\rangle = |R\rangle \alpha_R + |L\rangle \alpha_L$, where $\alpha_R = \langle R|\alpha\rangle$ and $\alpha_L = \langle L|\alpha\rangle$ can be regarded as wave functions. Suppose the particle can tunnel between the two potential wells; this tunneling effect is characterized by the Hamiltonian

$$\hat{H} = \Delta(|R\rangle\langle L| + |L\rangle\langle R|), \tag{1}$$

where Δ is a real number with the dimensions of energy.

(a) Find the normalized energy eigenkets and the corresponding energy eigenvalues.

(b) Suppose the system is represented by $|\alpha\rangle$ as given above at $t = 0$. Find the state vector $|\alpha(t)\rangle$ for $t > 0$ by applying the appropriate time evolution operator to $|\alpha\rangle$.

(c) If the particle is on the right side at $t = 0$, what is the probability of finding the particle on the left-hand side after a time t?

(d) Write down the coupled Schrödinger equations for the wavefunctions $\alpha_L(t) = \langle L|\alpha(t)\rangle$ and $\alpha_R(t) = \langle R|\alpha(t)\rangle$. Show that the solutions to these equations are just what you expect from the answer to (b).

(e) Suppose the printer made an error, and wrote H as

$$H = \Delta |R\rangle\langle L|. \tag{2}$$

By *explicitly* solving the most general time-evolution problem with this Hamiltonian, show that probability conservation is violated.

iii. A particle of mass m lies in a potential well where

$$V(x) = \begin{cases} \frac{1}{2} m\omega_0^2 x^2, & (x > 0) \\ \infty & (x \leq 0) \end{cases} \tag{3}$$

What is the energy and expected position of the particle in the ground-state? (Hint: think of how you can use wavefunctions from the harmonic oscillator problem to solve this problem.)