

Physics 228, Lecture 10
Thursday, February 24, 2005

Quantum Mechanics. Ch 38:8, 38:1-2, 38:7, 38:3-4

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We have just discussed one of two great revolutions in physics which occurred at the beginning of the 20'th century, the special relativity theory of Einstein, which proposed basic changes to the notions of space and time upon which classical mechanics was built. This change was not noticeable for objects moving much slower than light, but greatly changed the behavior of mechanics at speeds comparable to c . The other breakdown of classical physics had to do with behavior at short distances of light particles. This is quantum mechanics, and philosophically it constitutes a far greater break with classical theory than Einstein's.

Unlike relativity, which was basically the invention of one man, though building on the ideas of many, quantum mechanics was developed slowly by the work of many people.

1 Blackbody radiation and Planck

The first glimpse of quantum mechanics, historically, was an observation by Planck in 1900 concerning black body radiation. What is a black body? The blackest body is a hole in a light-tight container. Don't believe me? Let's see. Is it really black? Any light that enters has a hard time finding the small hole to get out again. But what if the walls emit light?

Demo box.
No reflection.
Emission?

All hot bodies have their atoms in motion, and if charged particles are oscillating, as we would expect of nuclei in a crystal, for example, electromagnetic radiation should be, and is, produced. The radiation emitted by hot bodies has two components. One, called the discrete spectrum, consists of discrete wavelengths dependent on the composition, as is manifest in the emission of light from neon bulbs, for example, or, as we mentioned, in the doublet of sodium vapor. There is also a continuous spectrum which is much less dependent on substance. In fact, it is based on what is called the black-body spectrum, which is the spectrum radiated, for example, by a hole in

the wall of a hot oven. From such a hole, there will be power radiated at all wavelengths, so the power is best discussed in terms of the rate at which energy is radiated in wavelengths in an infinitesimal interval $[\lambda, \lambda + d\lambda]$ per unit area of hole is $I(\lambda, T)d\lambda$, where $I(\lambda, T)$ is called the blackbody radiation spectrum. Classical treatment of the thermodynamics gave a prediction called the Rayleigh-Jeans law,

$$I(\lambda, T) = \frac{2\pi ck_B T}{\lambda^4} \quad \text{Rayleigh-Jeans classical prediction.}$$

Demo hot filament, brightens.

This law works well for long wavelengths, but it can't possibly be correct for short wavelengths, because if we ask about the total rate at which energy is radiated per unit area, we need to integrate over all wavelengths,

$$I = \int_0^\infty I(\lambda, T)d\lambda = 2\pi ck_B T \int_0^\infty \frac{d\lambda}{\lambda^4} \quad \text{diverges!}$$

where the divergence comes from the short wavelengths, $\lambda \rightarrow 0$. Physicists often call the short wavelength limit ultraviolet and the long wavelength limit infrared, even though those words properly refer to finite nonzero wavelengths, so this problem was called the ultraviolet catastrophe. It can't be right, because then hot charcoal would cool off instantly, which it doesn't.

What is actually observed is that the spectrum of a hot body reaches a maximum, and for a black body that maximum is reached at a wavelength given by

Wien's displacement law

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{m} \cdot \text{K}$$

First demo hot filament.

—
S&B 40.3
Y&F 38.31

$3 \frac{1}{4}'' \times 4 \frac{3}{4}''$

where T is the absolute temperature measured in Kelvin.

Planck was trying to understand the blackbody spectrum when he observed that he could get a very good fit to experimental results with the formula

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5 \left(e^{hc/\lambda k_B T} - 1 \right)} \quad \text{Planck's law.}$$

h is a constant which fits the data, for all black bodies, with the value

$$h = 6.626 \times 10^{-34} \text{J} \cdot \text{s.}$$

This constant is now known as Planck's constant. He also invented an explanation, based on classical physics plus an arbitrary assumption: oscillations

of frequency f could carry energy only in quanta, that is, the energy of any oscillator of frequency f could only be an integral number of multiples of hf . This assumption was completely ad-hoc — it had no justification except to explain this formula which fit the data, and even Planck did not believe the explanation.

With Planck's law, we can calculate the total power radiated per unit area,

$$I = \int_0^\infty \frac{2\pi hc^2}{\lambda^5 (e^{hc/\lambda k_B T} - 1)} d\lambda = 2\pi hc^2 \left(\frac{k_B T}{hc} \right)^4 \int \frac{x^3 dx}{e^x - 1} = 2\pi \int \frac{x^3 dx}{e^x - 1} \frac{k_B^4}{h^3 c^2} T^4.$$

The integral is $\pi^4/15$, so

$$I = \sigma T^4, \quad \text{where } \sigma = \frac{2\pi^5 k_B^4}{15h^3 c^2} = 5.67 \times 10^{-8} \text{W/m}^2 \cdot \text{K}^4$$

is known as Stefan's constant, and $I = \sigma T^4$ as Stefan's law.

2 Photoelectric Effect

Electrons in a metal are free to move around, as we know from the ease with which currents can be made in conductors. But under normal conditions they are not free to leave the metal. There is an attractive force which keeps them confined, with a certain energy required to extract them from the metal.

If one heats the metal to several thousand degrees, the thermal motion will give some electrons enough energy to leave the metal. This is what happens at the filament of a vacuum tube, such as at the back of the electron gun in your TV set, unless you have a modern plasma or LCD display. But another way electrons can get enough energy to leave the metal is to be hit by light. This is called the photoelectric effect.

Demo electro-
scope, white,
UV

Now each electron will need to be given enough energy, called ϕ , to overcome the attractive potential. If it is given an energy U which is more than ϕ , it can emerge from the surface of the metal with a kinetic energy of $K = U - \phi$. We can detect these emitted electrons if we gather them up by

applying a positive voltage to a collector C and measuring the current with the ammeter A . We would expect, with an attractive voltage, to collect many of the emitted electrons. But as a minimum energy ϕ is needed to get the electrons out of the metal, we expect no current in the dark. If we shine a light on the metal, we would expect that if the light is strong enough, electrons will get enough energy and some current will be detected. If the light is less intense, we might expect to have to wait a while while the oscillating field built up bigger oscillations, and hence more energy, in each electron.

S&B 40.6
 \sim Y&F 38.3
 $3'' \times 6''$

The surprise is that whether we get any current does not depend on the intensity of the light but on the color of the light. If there is a current, its magnitude will depend on the intensity, but only light of sufficiently high frequency will produce any current at all. Then even faint light of high frequency will produce some current. And there is never a delay — if the light is going to produce current it starts doing so right away.

One can get more information by reversing the voltage on the collector, making it negative with respect to the emitter, so that only electrons that have a sufficient kinetic energy K upon emission to overcome the repulsive potential, $K > e|V|$, can make it to the collector and produce a current. Adjusting the voltage until the current stops measures the maximum kinetic energy K_{\max} of the emitted electrons. And it turns out that this energy is independent of the intensity of the light but depends on its frequency with the simple relation:

$$K_{\max} = hf - \phi,$$

where h is Planck's constant, and ϕ , which is called the **work function** of the metal, is a different constant for each metal but is independent of any property of the light.

This has a simple but unexpected explanation. Each electron emitted by incident light of frequency f absorbs an energy hf regardless of what the intensity of the light is. If the light is more intense, more electrons will absorb this energy, but each one will not absorb more. This is an indication that a light wave actually consists of many **quanta**, each with energy hf , and that an interaction of an electron with the incident light is actually with an individual quantum of the light.

This explanation was given by Einstein in 1905, the same year he published his theory of special relativity and also a seminal paper on Brownian motion. It was for this work, not for relativity, that he won the Nobel Prize.

3 Compton Effect

Another effect which involved the interaction of light with individual electrons and did not seem to fit the classical picture is the Compton effect. When a free electron is exposed to an electromagnetic wave, classically we would expect it to oscillate and pick up momentum in the direction of the wave. Then, being an oscillating electric charge, we would expect it to radiate a new wave. The emitted wave should have a different frequency due to the doppler shift, because the electron would have picked up some velocity from the incident wave. The radiation should be in many directions with a spread of wavelengths, depending on how much velocity the electron had picked up.

This is not what is observed, and as we already know from the photoelectric effect that the interaction of an electron with a light wave doesn't behave classically, that should not surprize us. We can understand the results observed by considering the Compton effect to be the scattering of an electron, initially at rest, with a photon, a quantum of (usually X-ray) light, with energy $E = hf_0$ and momentum $p = hf_0/c$, to produce an electron with some momentum p_e and velocity v_e at the angle ϕ with respect to the incident X-ray beam, with energy $m_e c^2 \gamma$, together with a photon of frequency f' , energy $E' = hf'$ and momentum $p' = hf'/c$ at an angle θ as shown.

S&B 40.10
 (FP) briefly,
 then
 S&B 40.12
 =Y&F 38.27
 (40:12: 7 1/8" × 3
 3/4")

Energy conservation tells us

$$hf_0 + m_e c^2 = hf' + m_e c^2 \gamma, \quad \text{with } \gamma = 1/\sqrt{1 - v_e^2/c^2},$$

and momentum conservation says

$$P_x = \frac{hf_0}{c} = \frac{hf'}{c} \cos \theta + m_e \gamma v_e \cos \phi$$

$$P_y = 0 = -\frac{hf'}{c} \sin \theta + m_e \gamma v_e \sin \phi.$$

These are three equations which partially determine the results of the collision, which involve two angles θ and ϕ , a final frequency f' , and a final velocity v_e . Three equations in four variables means the outcome is not completely determined. As is typical in a scattering, the angle of one of the final particles is undetermined, but specifying that specifies the rest. So if we make

the angle at which we observe the outgoing radiation θ the free parameter, the rest of the final state is determined in terms of the initial state. (f_0 and the physical constants h , c and m_e). The results are usually stated in terms of the initial and final wavelengths instead of frequencies; these are simply related by $f = c/\lambda$, $f' = c/\lambda'$. The result¹ for the change in wavelength is

$$\Delta\lambda := \lambda' - \lambda_0 = \frac{hc}{m_e}(1 - \cos\theta).$$

Note that the outgoing wavelength is always longer than the incoming, be-

¹Here is the derivation, for completeness:

$$hf_0 = hf' + m_e c^2(\gamma - 1), \quad (1)$$

$$hf_0 = hf' \cos\theta + m_e c \gamma v_e \cos\phi, \quad (2)$$

$$hf' \sin\theta = m_e c \gamma v_e \sin\phi. \quad (3)$$

Move the f' terms to the left hand sides and square:

$$h^2(f_0^2 - 2f_0f' + f'^2) = m_e^2 c^4(\gamma^2 - 2\gamma + 1), \quad (4)$$

$$h^2(f_0 - f' \cos\theta)^2 = m_e^2 c^2 \gamma^2 v_e^2 \cos^2\phi, \quad (5)$$

$$hf' \sin^2\theta = m_e^2 c^2 \gamma^2 v_e^2 \sin^2\phi. \quad (6)$$

Add the last two equations together:

$$\begin{aligned} m_e^2 c^2 \gamma^2 v_e^2 (\cos^2\phi + \sin^2\phi) = m_e^2 c^2 \gamma^2 v_e^2 &= h^2(f_0^2 - 2f_0f' \cos\theta + f'^2 \cos^2\theta) + h^2 f'^2 \sin^2\theta \\ &= h^2(f_0^2 - 2f_0f' \cos\theta + f'^2). \end{aligned}$$

Turn this around and subtract (4) from it:

$$2h^2 f_0 f' (1 - \cos\theta) = m_e^2 c^2 (\gamma^2 v_e^2 - \gamma^2 + 2\gamma - 1).$$

Note that

$$\gamma^2 v_e^2 = \frac{v_e^2}{1 - v_e^2/c^2} = \frac{1}{1 - v_e^2/c^2} - 1 = \gamma^2 - 1,$$

so

$$2h^2 f_0 f' (1 - \cos\theta) = m_e^2 c^2 (2\gamma - 2) = 2m_e h(f_0 - f'),$$

where the last step is from Eq. 1. Finally, divide by $2h^2 f_0 f'$ and substitute c/λ_0 for f_0 and c/λ' for f'

$$1 - \cos\theta = \frac{m_e}{h} \left(\frac{1}{f'} - \frac{1}{f_0} \right) = \frac{m_e}{hc} (\lambda' - \lambda_0),$$

or

$$\Delta\lambda := \lambda' - \lambda_0 = \frac{hc}{m_e} (1 - \cos\theta).$$

cause the photon must lose some energy to the electron, which was initially at rest.

4 Early Atomic Models

While the idea that matter is made of atoms goes back to ancient Greece, it was only in the nineteenth century that some real information about what constituted an atom started to be developed, based on chemical observations of how atoms combined to form molecules, especially gases, and on properties and classification of the elements. J. J. Thomson discovered the electron, and that an atom consisted of a number of electrons together with some positive matter so as to make a whole which is electrically neutral. His model was that the positive charge was distributed in a big sphere in which the electrons were embedded, like plums in a plum pudding (well, in 19th century England, anyway).

This model was blown away by an experiment by Ernest Rutherford, in which alpha particles (now known to be the nuclei of helium atoms) were scattered off metal foils. Some of these underwent hard scattering (that is, their momentum was changed by a lot), scattering backwards instead of just deflected through a small angle. That could only be explained if the positive charge of the metal atoms were concentrated in a very small region, much smaller than the atom as a whole. We now know that the positive charge is in the nucleus, which is of the order of 10^{-15} m in radius, while the atom is of the order 10^{-10} m. Thus Rutherford proposed his model of the atom, in which the electrons revolved around the nucleus in orbits, much like the planets around the sun.

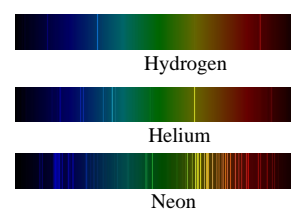
S&B 42.2, Y&F
38.15; Rutherford
scat and atom
7" × 3 3/4"

5 Atomic Spectra

A more familiar way in which electrons interact with light is in the emission of light by atoms. In the nineteenth century it was thought that atoms had electrons enmeshed in a large nuclear pudding, but with the Rutherford experiment of 1911 it became established that the positive nucleus has very small, and the electrons were outside the nucleus and presumably rotating around it as planets do the sun, constantly accelerating towards the

nucleus. But accelerating charges are supposed to radiate electromagnetic waves. Classically we would expect the electrons in atoms to radiate and lose energy, moving closer to the nucleus and revolving more rapidly, therefore radiating ever more rapidly and at higher frequency. This has two problems. First, it would imply that atoms are unstable — the electrons should fall into the center within a microsecond, which clearly doesn't happen.

Second, while atoms do radiate light when they are excited, they do not do so with a continuum of frequencies, as the classical Rutherford model would suggest, but instead with discrete spectra. That is, light comes out with only certain discrete wavelengths. For example, from hydrogen, the only wavelengths radiated in the visible region have wavelengths of 410.2, 434.1, 486.1, or 656.3 nm. Different atoms emit light at different frequencies, but each one has its discrete set of wavelengths.



I want to show you this in action, but first we need to be able to separate the different colors of light. You have each, I hope, picked up **one** small diffraction grating. First, try to see the spectrum of the white light by looking through the grating, not directly at the light but at a small angle to the side of the light. When you see rainbow colors you've got it — if not, make sure to hold it with the long side horizontal.

Now that you know how to disperse the colors with the grating, let's look at the spectra of several gases. Note that there are a few, or many, individual frequencies emitted, but we do not have a continuous spectrum the way we did with the white incandescent bulb. These emission lines are due to atoms dropping from an excited state to a lower energy state, giving off a photon in the process. You should see spectra that look a lot like what I showed on the transparency.

Atoms can also absorb light, and if a cool gas of atoms is in front of a source of a continuum of wavelengths, such as white light, only the light at the same discrete set of frequencies will be absorbed. The particular wavelengths at which light has been absorbed then appear as dark lines in a spectroscope's image. It was by observing the absorption lines from the solar atmosphere that helium was first discovered, which is how it got its name.

The simplest spectrum is that of hydrogen. All the emitted lines from hydrogen fit a formula

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right),$$

S&B 40.17

Y&F 38.9

 $2 \ 3/4^n \times 4^n$

where the

$$\text{Rydberg constant } R_H = 1.097\,373\,2 \times 10^7 \text{ m}^{-1},$$

and n_i and $n_f < n_i$ are positive integers. Those with a fixed value of n_f and different n_i are called a series, with the visible lines all in the **Balmer series**, which has $n_f = 2$.

6 Energy levels

We see that the discrete radiation that comes from a hydrogen atom seem to have wavelengths determined by two numbers, with $\frac{1}{\lambda} = f(n_f) - f(n_i)$. We also know from the photoelectric effect that a photon of light with wavelength λ has an energy $E_\lambda = hc\frac{1}{\lambda}$. This suggests that the photons radiated from an atom may get their energy by having the atom jump from one energy state to a state of lower energy, giving up the extra energy to the photon. Thus the energies $hcf(n)$ should represent a possible energy state of an atom, and only certain energies are possible states. This has no explanation in classical physics, but, as we shall see next time, is the basis of the Bohr model of the hydrogen atom. Other atoms also have discrete energy spectra, though not as simply described as hydrogen's.

The lowest energy state of an atom (or any other system) is called the **ground state**, and states with higher energies are called **excited states**. If an electron gun shoots electrons with definite energies into a gas of atoms, when the electron hits an atom it might give up its energy, or part of it, to the atom and move the atom into an excited state. But it can only do that if it has enough energy to reach that excited state. In 1914, Franck and Hertz sent electrons into a gas of mercury vapor. When the electrons had an energy less than 4.9 eV, nothing happened, but if the energy was higher than 4.9 eV, ultraviolet light with $\lambda = 250$ nm was emitted, which corresponds to a photon energy of 4.9 eV. This can be explained if we assume that mercury has an excited state 4.9 eV above the ground states. Then the sufficiently

energetic electrons can bump the atoms into that state, and later the atoms in that state can radiate an ultraviolet photon, returning to the ground state.

If we look at the sun, the spectrum is roughly like a black body of temperature 6000° C, but there are certain wavelengths missing - black lines appear in the spectrum. This is because light at that particular energy is absorbed by cooler gas in the sun's outer atmosphere, making transitions from the lower energy states to higher energy states while absorbing the photon. These are at the same wavelengths where the excited gas would have emitted light. From this **absorption spectrum**, an atomic spectrum never seen before was observed, and this was the discovery of a new element called helium (helios is the Greek name for the sun). Of course subsequently helium was discovered on Earth as well, and now is used to fill birthday balloons.

The emission spectrum of hydrogen and many other atoms were well explored in the middle-late nineteenth century, but there was no explanation of why atoms radiated at these frequencies. The answer was found by Bohr in 1913. But we will wait until next time to find what that answer is.

7 Summary

- The blackbody spectrum, which gives the power per unit area per unit wavelength radiated by a black body, is given by

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5 (e^{hc/\lambda k_B T} - 1)}.$$

This form cannot be understood classically, but was “explained” by an *ad-hoc* model of Planck in which he assumed that the energy of oscillators is quantized.

- The constant of nature h in this empirical law is known as Planck's constant,

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}.$$

- The maximum of this intensity as a function of wavelength is at $\lambda_{\text{max}} = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$. This is called **Wien's displacement law**.
- When electrons are emitted from a metal by photoemission from light of frequency f , they emerge with a maximum kinetic energy of

$$K_{\text{max}} = hf - \phi,$$

where ϕ depends on the metal and is called the **work function** of that metal. The intensity effects the rate at which electrons are emitted but not whether they are emitted nor what the maximum energy at which they are emitted is.

- The Compton effect describes the scattering of an electron by light. It can be understood as a scattering with a photon, at which the final photon has a longer wavelength than the incident photon, with

$$\Delta\lambda := \lambda' - \lambda_0 = \frac{hc}{m_e}(1 - \cos\theta),$$

where θ is the angle of scatter of the photon.

- Atoms radiate light only at certain characteristic frequencies. For hydrogen,

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \quad \text{with } R_H = 1.097\,373\,2 \times 10^7 \text{m}^{-1},$$

and n_i and $n_f < n_i$ are positive integers. The “lines” or wavelengths with $n_f = 2$ are called the **Balmer series**.

- The Franck-Hertz experiment found that electrons with energy larger than 4.9 eV could excite mercury vapor and have it subsequently emit ultraviolet light, while if the individual electrons has a lower energy they produced no ultraviolet light. This is understood in terms of mercury having an excited state at 4.9 eV above the ground state.
- Generally the discrete spectra could be understood as transitions between atomic states of specific possible energy levels, though classical mechanics cannot explain why there should only be specific energies at which at atom can be found.