Lecture 22 Monday April 22

L-R-C circuits
AC circuits
This week

- Homework due Thursday April 25, 11:59 pm
- Recitation quiz: HW9,L20+21 (Chapter 29); activity Chapter 31

**Required on-line survey** = 1% of your grade

Take survey TODAY (if not already) – deadline May 1

https://rutgers.ca1.qualtrics.com/jfe/form/SV_3w1DABbPtzNxGzX

on your attitudes toward physics and textbook use. Taking the survey seriously and completing it will give you full credit for 1% of your grade. You can opt out of having your data used in the research, but you must complete the survey.

**Final exam: Friday, May 10, 2019 4:00 to 7:00 PM in PLH**

Request conflict exam by April 26: If you have 3 exams on May 10 OR 3 exams in a row, including Physics 227, contact Professor Cizewski with your entire exam schedule.

30 multiple choice questions: ≈17 from Chapters 29-32, ≈13 cumulative Chapters 21-28. All exams are closed-book, **no calculators or other electronic devices allowed.**
Final Exam Details

**Final exam**

- Friday, May 10, 2019 in Physics Lecture Hall: 4:00 – 7:00 pm
- 30 multiple choice questions:
  - \(\approx 17\) from Chapters 29-32, \(\approx 13\) from Chapters 21-28
- Exam review Thursday, May 9

All exams are closed-book, **no calculators or other electronic devices allowed**. All questions will be multiple choice. For the final exam, you may bring with you three (3) "formula sheets", on 8.5 x 11 inch sheets of paper (OK to use both sides) on which you may **hand write** any formulae or diagrams or notes or problem solutions that might be helpful to you during the exam. Information on the sheets must be handwritten, no attachments are allowed. The numerical values of relevant constants will be provided to you. You should bring #2 pencils to the exams for the computer forms.

**Types of questions**: Like I-clickers, simple numbers, formulae

**Study**: Homework, I clickers, examples in textbook, collaborative+pre-REC


**Free tutoring via MSLC before May 6**: https://rlc.rutgers.edu/services/peer-tutoring
• Changing magnetic flux in loop with time-dependent current $i(t)$
  • Back EMF, that depends upon inductance (geometry) of the loop
  • Self inductance of one loop and mutual inductance of two loops

• Inductor is a circuit element where $V_{ab}$

• Inductors store energy of magnetic field

• L-R circuits
  • Growth and decay of $i(t)$ is exponential

• L-C circuits
  • Electric and magnetic energy exchanged between capacitor and inductor

• L-R-C circuit
  • Damped oscillations

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L \frac{di}{dt} = -M \frac{di}{dt}$$

$$L \equiv \frac{N\Phi_B}{i}, \quad M \equiv \frac{N_2\Phi_{1\rightarrow2}}{i}$$

$$V_{ab} = V_a - V_b = L \frac{di}{dt}$$

$$U = \frac{1}{2}LI^2, \quad u_B = \frac{B^2}{2\mu_0}$$

$$\tau = \frac{L}{R}, \quad \omega = \sqrt{\frac{1}{LC}}$$
Kirchhoff’s loop equation after C charge, switch S closed

\[ 0 = -iR - L \frac{di}{dt} - \frac{q(t)}{C} \]

\[ 0 = -R \frac{dq(t)}{dt} - L \frac{d^2 q(t)}{dt^2} - \frac{q(t)}{C} \]

\[ \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{LC} q(t) = 0 \]

Solution for weak damping: \( \frac{1}{LC} \gg \frac{R^2}{4L^2} \)

\[ q(t) = Q_m \exp\left(-\frac{Rt}{2L}\right) \cos(\omega't + \phi_0) \]
Circuits with L-R-C: Damping of oscillations

Kirchhoff’s loop equation

\[ 0 = -iR - L \frac{di}{dt} - \frac{q(t)}{C} = -R \frac{dq(t)}{dt} - L \frac{d^2 q(t)}{dt^2} - \frac{q(t)}{C} \]

\[ \frac{d^2 q(t)}{dt^2} + \frac{R}{L} \frac{dq(t)}{dt} + \frac{1}{LC} q(t) = 0 \]

Solution for weak (under) damping: \( \frac{1}{LC} \gg \frac{R^2}{4L^2} \)

\[ q(t) = Q_m \exp\left(-\frac{Rt}{2L}\right) \cos(\omega't + \phi_0) \]

\[ \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

Cosine function
Amplitude decreases exponentially
Circuits with L-R-C: Damping of oscillations

Solution for weak (under) damping: \( \frac{1}{LC} \gg \frac{R^2}{4L^2} \)

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Solution for critical damping, R large

Solution for strong (over) damping, R very large
The switch S is closed at $t=0$. What is the current $I$ at time $t=1$ s?

A. $I = 0$ A
B. $I = 5$ A
C. $I = 10$ A
D. $I = 15$ A
E. $I = 20$ A
The switch S is closed at $t=0$. What is the current $I$ at time $t=1$ s?

A. $I = 0$ A  
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Summary Chapter 30

- Changing magnetic flux in loop with time-dependent current $i(t)$
  - Back EMF, that depends upon inductance (geometry) of the loop
  - Self inductance of one loop and mutual inductance of two loops
- Inductor is a circuit element where $V_{ab}$
- Inductors store energy of magnetic field
- L-R circuits
  - Growth and decay of $i(t)$ is exponential
- L-C circuits
  - Electric and magnetic energy exchanged between capacitor and inductor
- L-R-C circuit
  - Damped oscillations

Mathematical equations:

$$E = -\frac{d\Phi_B}{dt} = -L \frac{di}{dt} = -M \frac{di}{dt}$$

$$L \equiv \frac{N\Phi_B}{i} \quad M \equiv \frac{N_2\Phi_{1\rightarrow2}}{i}$$

$$V_{ab} = V_a - V_b = L \frac{di}{dt}$$

$$U = \frac{1}{2} LI^2 \quad u_B = B^2 \sqrt{\frac{2}{2\mu_0}}$$

$$\tau = \frac{L}{R}$$

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$
Chap 31: AC circuits
Chapter 31 outline

• Analyze $L-R-C$ series circuit
  • With sinusoidal EMFs and with different frequencies
    • Alternating current (AC)
  • Use reactance to describe voltage across a circuit element w/ AC

• Tool: phasors to describe sinusoidally varying quantities.

• Calculate power flowing into or out of AC circuit

• Role of transformers
AC circuits

Power source that provides an alternating voltage

\[ v = V_{max} \cos \omega t \]

- Where \( V_{max} \) is the maximum output of the voltage source or voltage amplitude.
- Angular frequency \( \omega \) related to frequency \( f \) and period \( T \)

\[ \omega = 2\pi f = \frac{2\pi}{T} \]

Sources:
- Generators we discussed in Chap 29
- Electrical oscillators
\[ v = V_{\text{max}} \cos \omega t \] Resistors in an AC circuit

Simple circuit with resistor R and AC power source

\[ \nu - i_R R = 0 \]

\[ i_R = \frac{\nu}{R} = \frac{V_{\text{max}}}{R} \cos \omega t = I_{\text{max}} \cos \omega t \]

- Instantaneous voltage \( v_R \) across resistor

\[ v_R = I_{\text{max}} R \cos \omega t \]

\[ V_R = I_{\text{max}} R \]

\[ v_R = V_R \cos \omega t \]

Both \( i_R \) and \( v_R \) are cosine functions

\[ \triangleright \text{they are in phase} \]
At $t=0$
\[ i_R = I_{\text{max}} \]
\[ v_R = V_R \]

Increase angle by $\omega t$

\[ i_R = I_{\text{max}} \cos \omega t \]
\[ v_R = V_R \cos \omega t \]
Current and voltage stay in phase

**Phasors**

- “vector” where magnitude given by maximum current/voltage
- “direction” is relative angle/phase from where started
- Magnitude of instantaneous current/voltage is projection on x axis

\[
i_R = I_{\text{max}} \cos \omega t \\
u_R = V_R \cos \omega t
\]
Inductors in an AC circuit

Simple circuit with inductor $L$ and AC power source

$$v_L = L \frac{di}{dt} = L \frac{d}{dt} \left( I_{\text{max}} \cos \omega t \right) = -I_{\text{max}} \omega L \sin \omega t$$

- Can rewrite in terms of cosine
  - Cosine function with “head start” of 90°
  $$v_L = -I_{\text{max}} \omega L \sin \omega t = I_{\text{max}} \omega L \cos(\omega t + 90°)$$
- If define the phase of the voltage relative to the current
  $$i = I_{\text{max}} \cos \omega t$$
  $$v_L = I_{\text{max}} \omega L \cos(\omega t + \phi = 90°)$$
Inductors in an AC circuit

\[ i = I_{\text{max}} \cos \omega t \]

\[ v_L = I_{\text{max}} \omega L \cos(\omega t + \phi = 90^\circ) \]

If change angle of current by \( \omega t \)
Then \( v_L \) leads the current by 90°

Instantaneous values of the current and voltage are projections on x-axis.

Physical sense:
Inductors in an AC circuit

\[ i = I_{\text{max}} \cos \omega t \]

\[ v_L = I_{\text{max}} \omega L \cos(\omega t + \phi = 90°) \]

Physical sense:
When first close a switch
- Voltage drop across inductor proportional to \( \frac{di}{dt} \)
- Current in circuit = 0

Amplitude for \( v_L \)
Inductive reactance
Units: Ohms

\[ V_L = I_{\text{max}} \omega L \]

\[ X_L \equiv \omega L \]

\[ V_L = I_{\text{max}} X_L \]
Capacitors in an AC circuit

Simple circuit with capacitor C and AC power source

- DC circuit: \( V = Q / C \), so need to write \( \nu_C \) in terms of current \( i \)

\[
\nu_C = \frac{I_{\text{max}}}{\omega C} \sin \omega t = \frac{I_{\text{max}}}{\omega C} \cos(\omega t - \phi = 90^\circ)
\]

- Cosine function “lags behind” by 90°

- If define the phase of the voltage relative to the current

\[
i = I_{\text{max}} \cos \omega t
\]

\[
q = \frac{I_{\text{max}}}{\omega} \sin \omega t
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\[
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Capacitors in an AC circuit

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Capacitors in an AC circuit

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\[ v_C = \frac{I_{\text{max}}}{\omega C} \cos(\omega t - 90^\circ) \]

Amplitude for \( v_C \)
\[ V_C = \frac{I_{\text{max}}}{\omega C} \]

Capacitive reactance
Units: Ohms
\[ X_C \equiv \frac{1}{\omega C} \]

Physical sense:
When first close the switch, \( v_c=0 \), but current flows
Need to wait for capacitor to be charged
Comparing ac circuit elements

<table>
<thead>
<tr>
<th>Circuit element</th>
<th>Amplitude relationship</th>
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<td>Resistor</td>
<td>$V_R = I_{\text{max}}R$</td>
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An inductor is connected across an ac source as shown. For this circuit, what is the relationship between the instantaneous current \( i \) through the inductor and the instantaneous voltage \( v_{ab} \) across the inductor?

A. \( i \) is maximum at the same time as \( v_{ab} \).

B. \( i \) is maximum one-quarter cycle before \( v_{ab} \).

C. \( i \) is maximum one-quarter cycle after \( v_{ab} \).

D. Two of A, B, and C are possible, depending on circumstances.

E. All three of A, B, and C are possible, depending on circumstances.
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D. Two of A, B, and C are possible, depending on circumstances.

E. All three of A, B, and C are possible, depending on circumstances.
L-R-C series circuit

L-R-C in series

- Current at any time is same for any point in circuit.
  \[ i = I_{\text{max}} \cos \omega t \]

- However, voltages across the circuit elements will not be the same.

- \( v_R \): in phase with current \( i \)
  \[ v_R = I_{\text{max}} R \cos \omega t \]

- \( v_L \): leads current by 90°
  \[ v_L = I_{\text{max}} X_L \cos(\omega t + 90°) \]

- \( v_C \): lags current by 90°
  \[ v_C = I_{\text{max}} X_C \cos(\omega t - 90°) \]

To find magnitude of voltage add phasors like vectors
\[ v_R = I_{\text{max}} R \cos \omega t \]

\[ v_L = I_{\text{max}} X_L \cos (\omega t + 90^\circ) \]

\[ v_C = I_{\text{max}} X_C \cos (\omega t - 90^\circ) \]

\[ i = I_{\text{max}} \cos \omega t \]

\[ V = I_{\text{max}} \sqrt{R^2 + (X_L - X_C)^2} = I_{\text{max}} Z \]

Defines impedance \( Z \)

http://www.learnabout-electronics.org/ac_theory/lcr_series_91.php
**L-R-C series circuit**

\[ v_R = I_{\text{max}} R \cos \omega t \]
\[ v_L = I_{\text{max}} X_L \cos(\omega t + 90^\circ) \]
\[ v_C = I_{\text{max}} X_C \cos(\omega t - 90^\circ) \]
\[ v = V_{\text{max}} \cos(\omega t + \phi) \]

**Impedance Z**

Units: Ohms

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \]

\[ V_{\text{max}} = I_{\text{max}} \sqrt{R^2 + (X_L - X_C)^2} = I_{\text{max}} Z \]

\[ \tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R} \]
Root mean square values of $i$ and $v$

\begin{align*}
u_R &= I_{\text{max}} R \cos \omega t \\
v_L &= I_{\text{max}} X_L \cos (\omega t + 90^\circ) \\
v_C &= I_{\text{max}} X_C \cos (\omega t - 90^\circ) \\
v &= V_{\text{max}} \cos (\omega t + \phi)
\end{align*}

Average values of cosine function = 0
Not very practical
More practical: root-mean-square values
Root mean square values of $i$ and $v$

$v_R = I_{\text{max}} R \cos \omega t$
$v_L = I_{\text{max}} X_L \cos(\omega t + 90^\circ)$
$v_C = I_{\text{max}} X_C \cos(\omega t - 90^\circ)$
$v = V_{\text{max}} \cos(\omega t + \phi)$

Average values of cosine function $= 0$
Not very practical
More practical: root-mean-square values

\[
i^2 = I_{\text{max}}^2 \cos^2 \omega t = \frac{I_{\text{max}}^2}{2} (1 + \cos 2\omega t)
\]

\[
= \frac{I_{\text{max}}^2}{2} + \frac{I_{\text{max}}^2}{2} \cos \omega t
\]

Root-mean-square of $i$: \[I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}}\]
Root mean square values of $i$ and $v$

\[ v_R = I_{\text{max}} R \cos \omega t \]
\[ v_L = I_{\text{max}} X_L \cos (\omega t + 90^\circ) \]
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\[ v = V_{\text{max}} \cos (\omega t + \phi) \]

Root-mean-square of $i$:
\[ I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} \]

Root-mean-square of $v^2$:
\[ V_{\text{rms}} = \frac{V_{\text{max}}}{\sqrt{2}} \]

Example: “Household voltage = 120 V” is the $V_{\text{rms}}$.

Therefore $V_{\text{max}}$ for household voltage is
\[ V_{\text{max}} = \sqrt{2}V_{\text{rms}} = 170 \text{ V} \]
For the circuit shown, which condition gives the maximum voltage across the capacitor?

A. The maximum voltage across the capacitor occurs when \( X_C = \frac{1}{\omega C} = X_L = \omega L \)

B. The maximum voltage across the capacitor occurs when \( \omega = 0 \)

C. The maximum voltage across the capacitor occurs when \( \omega = \text{infinity} \).

D. The maximum voltage across the capacitor occurs when \( C/L \) is very, very large.

\[
Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}
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D. The maximum voltage across the capacitor occurs when \( C/L \) is very, very large.
Voltage, current, phase angle, reactance

\[ i = I_{\text{max}} \cos \omega t \]
\[ v = V_{\text{max}} \cos(\omega t + \phi) \]
\[ V_{\text{max}} = I_{\text{max}} Z \]

r.m.s. Values
\[ I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} \]
\[ V_{\text{rms}} = \frac{V_{\text{max}}}{\sqrt{2}} \]

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\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \]
\[ \tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R} \]

\[ V_L = I X_L \]
\[ V = I Z \]
\[ V_R = I R \]
\[ V_C = I X_C \]
Next lecture:
Continue AC Circuits + Maxwell’s Equations

Let there be light!