Mathematical Tripos Part III
Michaelmas Term 2003
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QUANTUM FIELD THEORY

Particle Scattering and Feynman Diagrams

1. A quantum field theory has a Lagrangian density $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$ where

$$\mathcal{L}_0(x) = \frac{1}{2}(\partial\phi(x))^2 - \frac{1}{2}m^2(\phi(x))^2 + \frac{1}{2}(\partial\psi(x))^2 - \frac{1}{2}\mu^2(\psi(x))^2 + \frac{1}{2}(\partial\chi(x))^2 - \frac{1}{2}M^2(\chi(x))^2$$

and

$$\mathcal{L}_I(x) = -g\phi(x)\psi(x)\chi(x)\,.$$

Use Wick's theorem to show that the $O(g^2)$ contribution to the scattering amplitude for the elastic scattering process $\phi(p_1) + \psi(p_2) \to \phi(p_3) + \psi(p_4)$ associated with the diagram

is

$$T = \frac{-g^2}{s - M^2}$$

where the dashed line represents the χ propagator, and s is the square of the centre-of-mass energy. Identify the remaining $O(g^2)$ contribution to the process by drawing the appropriate Feynman diagram and evaluate it.

2. Verify that the symmetry factor for the "figure of eight" Feynman diagram of ϕ^4 theory is $\frac{1}{8}$. What is the integral corresponding to the diagram below?

3. Verify in detail, to 2nd order in λ, that in ϕ^4 theory $\langle 0|S|0\rangle$ is given by the exponential of the sum of distinct vacuum bubble types. [*You may work diagrammatically, but you should calculate all the combinatorial factors.*]

4. By considering appropriate simple diagrams make a guess at the Feynman rules appropriate to a theory with the Lagrangian $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$ where

$$\mathcal{L}_0(x) = \frac{1}{2}\left(\partial\phi_\alpha(x)\right)^2 - \frac{1}{2}m^2\left(\phi_\alpha(x)\right)^2$$

and

$$\mathcal{L}_I(x) = -\frac{\lambda}{8}\left((\phi_\alpha(x))^2\right)^2 ,$$

where summation over $\alpha = 1,\ 2,\ 3$ is assumed.
Begin by showing that the Feynman propagator for the free field theory (i.e. $\lambda = 0$) is of the form

$$\langle 0|T\phi_\alpha(x)\phi_\beta(y)|0\rangle = \delta_{\alpha\beta}D_F(x-y)$$

where $D_F(x-y)$ is the standard function, then compute the scattering amplitude for

$$\phi_\alpha(p_1) + \phi_\beta(p_2) \to \phi_\gamma(p_3) + \phi_\epsilon(p_4)$$

to lowest order in λ .

5. Using the momentum space Feynman rules, write down the fermion loop integral occuring in the following diagrams in Yukawa theory and QED, respectively.

Simplify the Trace expressions that occur in the numerators of the integrals.
[*There is a detailed discussion of the second diagram in many QFT books, see e.g. Peskin and Schroeder Sect. 7.5, or Chang Sect. 6.2.*]

6. Any vector function $\mathbf{f}(\mathbf{x})$ has a decomposition into a sum of transverse (zero divergence) and longitudinal (zero curl) parts, namely

$$\mathbf{f} = \nabla\times\mathbf{g} + \nabla h \equiv \mathbf{f}^T + \mathbf{f}^L \qquad (*)$$

where $\mathbf{g}$ and h are unique if one imposes the additional constraint $\nabla\cdot\mathbf{g} = 0$ and certain vanishing conditions at infinity. By taking the divergence and curl of equation $(*)$, determine $\mathbf{g}$ and h in terms of $\mathbf{f}$.
Show formally that

$$\mathbf{f}^T = \mathbf{f} - \nabla(\nabla^2)^{-1}\nabla\cdot\mathbf{f}\,.$$

Use this result to comment on the commutation relations of the quantised electromagnetic gauge potential (in Coulomb gauge).

7. Consider the Compton scattering process $\gamma(k,\epsilon) + e(p,s) \to \gamma(k',\epsilon') + e(p',s')$, where γ and e denote photon and electron. Use the Feynman rules for QED to show that the contribution, $T_2^{(a)}$, to the scattering amplitude associated with the following

diagram

is

$$T_2^{(a)} = \bar{u}_{s'}(p')\epsilon'.\gamma\frac{1}{\gamma.(k+p)-m+i\epsilon}\gamma.\epsilon u_s(p)\,.$$

Compute the contribution, $T_2^{(b)}$, corresponding to the diagram

Form the complete $O(e^2)$ contribution $T_2 = T_2^{(a)} + T_2^{(b)}$. Show that if ϵ is replaced by k in the expression for T_2 the result vanishes. (*Recall the equation satisfied by* $u_s(p)$ *and deduce the equation for* $\bar{u}_s(p)$.) Check that the same holds true if ϵ' is replaced by k' . Explain why this means that the transversality constraint on the polarisation ϵ of an external photon of momentum k can be relaxed to the condition $k.\epsilon = 0$.

8. Check the following identities:

$$\mathrm{Tr}\gamma^\mu = 0 \qquad \mathrm{Tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$

$$\mathrm{Tr}\gamma.p = 0 \qquad \mathrm{Tr}\gamma.p\gamma.q = 4p.q$$

$$\mathrm{Tr}\gamma^\mu\gamma^\nu\gamma^\sigma = 0 \qquad \mathrm{Tr}\gamma.p\gamma^\nu\gamma.q = 0$$

$$\mathrm{Tr}\gamma.p\gamma^\mu\gamma.q\gamma^\nu = 4\left(p^\mu q^\nu + p^\nu q^\mu - p.qg^{\mu\nu}\right)$$

$$\sum_s u_s(p)\bar{u}_s(p) = \gamma.p + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \gamma.p - m\,.$$

9. Assume (or show) that

$$\left[\bar{u}_{s'}(p')\gamma^\mu u_s(p)\right]^* = \left[\bar{u}_s(p)\gamma^\mu u_{s'}(p')\right]\,.$$

Hence show that

$$\sum_{ss'}\left[\bar{u}_{s'}(p')\gamma^\nu u_s(p)\right]^*\left[\bar{u}_{s'}(p')\gamma^\mu u_s(p)\right] = \mathrm{Tr}\left[(\gamma.p'+m)\gamma^\mu(\gamma.p+m)\gamma^\nu\right]\,.$$

Similarly show that

$$\sum_{sr}\left[\bar v_r(q)\gamma^\nu u_s(p)\right]^*\left[\bar v_r(q)\gamma^\mu u_s(p)\right] = \mathrm{Tr}\left[(\gamma.q-m)\gamma^\mu(\gamma.p+m)\gamma^\nu\right]$$

$$= \mathrm{Tr}\gamma.q\gamma^\mu\gamma.p\gamma^\nu - m^2\mathrm{Tr}\gamma^\mu\gamma^\nu$$

$$= 4\left[p^\mu q^\nu + p^\nu q^\mu - (p.q+m^2)g^{\mu\nu}\right]\,.$$

10. Show from the Feynman rules for QED that the amplitude corresponding to the diagram for the scattering process $e^-(p,s)+e^+(q,r)\to\mu^-(p',s')+\mu^+(q',r')$

where μ denotes the muon, is

$$T = e^2\bar u_{s'}(p')\gamma^\mu v_{r'}(q')\frac{-1}{s}\bar v_r(q)\gamma_\mu u_s(p)\,.$$

Let m, M denote the electron and muon masses, respectively. Show that

$$\sum_{srs'r'}|T|^2 = \frac{e^4}{s^2}\mathrm{Tr}\left[(\gamma.p'+M)\gamma^\mu(\gamma.q'-M)\gamma^\nu\right]\mathrm{Tr}\left[(\gamma.p+m)\gamma_\nu(\gamma.q-m)\gamma_\mu\right]\,.$$

In order to simplify this assume that the momentum components are so large that it is a good approximation to neglect the electron and muon masses. With $m=M=0$ show that

$$\sum_{srs'r'}|T|^2 = \frac{e^4}{s^2}32\left[p.p'q.q' + p.q'q.p'\right]$$

$$= 4e^4(1+\cos^2\theta)$$

where θ is the scattering angle in the centre-of-mass frame.