Mathematical Tripos Part III — **Lent Term 2004**
Advanced Quantum Field Theory: Examples 4 — **Dr. J.M. Evans**

Gauge fixing and BRST

1. The Yang-Mills lagrangian $-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$ is modified by a gauge-fixing term $-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$. Derive the propagator in momentum space.

2. Consider a classical Yang-Mills gauge theory with fields A^a_μ. By a suitable gauge transformation we may always choose $A^a_0 = 0$ (temporal gauge). What are the residual gauge transformations which respect this condition? Show that if the gauge field is abelian and satisfies its classical equation of motion we may impose the further condition $\partial_i A_i = 0$. Is this true in the non-abelian case?

3. Consider the Faddeev-Popov construction applied to a Yang-Mills theory with gauge fixing conditions (of either 'delta-function' or 'Gaussian' type) involving a function $\mathcal{F}_a(A^a_\mu)$, as discussed in lectures. Show that the classical equation of motion obeyed by the ghost field c_a coincides with the condition that must be satisfied by the parameter of a residual gauge transformation, i.e. one which leaves $\mathcal{F}_a$ invariant.

4. The electromagnetic field can be quantized by using the Faddeev-Popov method to add a gauge-fixing term $-\frac{1}{2}\mathcal{F}^2$ to the lagrangian, where $\mathcal{F} = \partial_\mu A^\mu + A_\mu A^\mu$. Find the full action (including ghosts) and write down the corresponding Feynman rules.

5. Write down the BRST transformations for a gauge theory with Yang-Mills fields A^a_μ coupled minimally to fermions ψ_i transforming in the fundamental representation of some gauge group with generators $(T^a)_{ij}$. Check that the fermionic BRST operation s satisfies $s^2 = 0$ on all fields.

Some extra questions

6. Take the lagrangian for a Dirac fermion ψ of mass m and add interactions $\bar{\eta}\psi$ and $\bar{\psi}\eta$ where η and $\bar{\eta}$ are classical Grassmann sources. Evaluate $W[\eta, \bar{\eta}]$ by comparing with an appropriate result for finite-dimensional Grassmann integrals, showing that this leads to the standard expression for the Dirac propagator.

7. Find the form of the propagator for a Yang-Mills field in axial gauge $n^\mu A^a_\mu = 0$ (with $n^2 = -1$).

8. A bosonic field ϕ has interactions of type $\lambda\phi^3$ and $\lambda^2\phi^4$. Write down equations relating the number of loops, propagators, vertices (of each type) and external lines for a connected diagram. Use these relations to show that the perturbative series for the connected n-point function contains only even or only odd powers of λ. [Are the assumptions about the interactions dimensionally consistent? – Compare with Yang-Mills theories!]

9. Consider a scalar field theory with renormalized coupling $\lambda(\mu)$, mass $m(\mu)$ and field $\phi(\mu) = Z(\mu)^{-1/2}\phi_0$ where ϕ_0 is the μ-independent bare field (as usual, μ is the renormalization scale). Define

$$\beta = \mu\frac{d\lambda}{d\mu}\ , \qquad \rho = \mu\frac{dm}{d\mu}\ , \qquad \gamma = \mu\frac{d}{d\mu}\log Z^{1/2}$$

and note that for any function $f(\mu, \lambda, m)$ we have

$$\mu\frac{df}{d\mu} = \mu\frac{\partial f}{\partial\mu} + \beta\frac{\partial f}{\partial\lambda} + \rho\frac{\partial f}{\partial m}$$

The bare correlation functions $\tau_0^{(n)}(p_1, \ldots, p_n)$ depend on the cut-off but not on μ. They are related to the finite, renormalized correlation functions $\tau^{(n)}(p_1, \ldots, p_n)$ which are functions of μ, λ and m, by $\tau_0^{(n)} = Z^{n/2}\tau^{(n)}$. Show that these facts lead immediately to the *renormalization group equation for correlation functions*:

$$\left[\ \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \rho\frac{\partial}{\partial m} + n\gamma\ \right]\tau^{(n)}(p_1, \ldots, p_n)$$

Now consider the 2-point function or full propagator $\tau^{(2)}(p) = \tau^{(2)}(p, -p)$ and suppose

$$\tau^{(2)}(p) = \frac{iR^2}{p^2 - m_*^2}\ +\ \text{(regular function of } p)$$

which identifies m_* as the physical mass. Show that m_* is independent of μ and that

$$\left[\ \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \rho\frac{\partial}{\partial m} + \gamma\ \right]R = 0$$

[Hint: take a μ derivative and compare coefficients of poles terms.] Deduce that the combination of correlation functions appearing in S-matrix elements:

$$R^n\,\frac{\tau^{(n)}(p_1, \ldots, p_n)}{\tau^{(2)}(p_1)\ldots\tau^{(2)}(p_n)}$$

is indeed independent of μ, as required for a physical quantity.