Mathematical Tripos Part III **Lent Term 2004**
Advanced Quantum Field Theory: Examples 2 **Dr. J.M. Evans**

Functional Methods and Feynman Graphs

1. Write down the generating functional for a free scalar field $\phi(x)$ of mass m in the presence of a classical source $J(x)$. Calculate the four-point correlation function by taking successive functional derivatives. Depict the result using Feynman graphs.

2. Given $Z[J]/Z[0] = \exp iW[J]$, find all functional derivatives of $Z[J]$ of order four or less in terms of those of $W[J]$. Assuming $\delta W/\delta J(x) = 0$ when $J = 0$, find an expression for the full four-point function of any field theory in terms of connected correlation functions.

3. Use the position space Feynman rules to write down an expression for the graph

in ϕ^4 theory. Use this to derive the corresponding result in momentum space (following the steps outlined in lectures).

4. Prove the identity

$$G(\partial/\partial b)F(b) = F(\partial/\partial u)G(u)e^{ub}|_{u=0}$$

by assuming the functions F and G are expandable as power series (by linearity, it then suffices to consider $F(x) = x^n$ and $G(x) = x^m$). Extend this to the case of many variables given in the notes.

5. Derive the relationship between the number of loops, the number of powers of the coupling λ, and the number of external lines for any Feynman diagram in the theory of a scalar field with interaction $\lambda\phi^3$. Draw all connected graphs contributing to the propagator (two-point function) up to two loops. Find as many graphs as you can contributing to the 3-point and 4-point correlation functions to two loops.

6. Starting from the definition of the effective action $\Gamma[\phi]$ in terms of the generating functional for connected graphs $W[J]$, relate $\delta^2\Gamma/\delta\phi(x_1)\delta\phi(x_2)$ to $\delta^2 W/\delta J(y_1)\delta J(y_2)$ as done in lectures. Now make further use of the functional chain rule to show that $\delta^3\Gamma/\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)$ can be written as

$$\int dy_1 \int dy_2 \int dy_3 \, \frac{\delta^3 W}{\delta J(y_1)\delta J(y_2)\delta J(y_3)} \, \frac{\delta^2\Gamma}{\delta\phi(x_1)\delta\phi(y_1)} \, \frac{\delta^2\Gamma}{\delta\phi(x_2)\delta\phi(y_2)} \, \frac{\delta^2\Gamma}{\delta\phi(x_3)\delta\phi(y_3)}$$

up to a numerical coefficient, which you should determine. Interpret this graphically to show the expression represents the (amputated) 1PI 3-point function.

7. Recall how the generating functional $Z[J]$ of an interacting theory can be written in terms of the generating functional $Z_0[J]$ of a free theory. Using an explicit formula for $Z_0[J]$ for a free scalar field and taking the interaction lagrangian $\mathcal{L}_I(\phi) = -\lambda\phi^4/4!$, obtain the precise numerical coefficients of the terms in the resulting expansion corresponding to each of the Feynman diagrams

Check that each coefficient is consistent with the rule for symmetry factors of graphs given in lectures. [Do not attempt to *evaluate* the graphs.]

Gaussian Integrals and Grassmann Variables

8. Let z_j be n complex variables and A_{ij} a positive-definite hermitian matrix. Show that

$$\int dz_1 dz_1^* \ldots dz_n dz_n^* \, \exp(-z_i^* A_{ij} z_j) = (const)(\det A)^{-1}$$

(where, by definition, $dz dz^* = dx dy$ if $z = x + iy$).

9. Show that

$$\int d\omega_1 \ldots d\omega_{2n} \, \exp(-\tfrac{1}{2}\omega_i A_{ij} \omega_j + \omega_j \eta_j) = (\det A)^{1/2} \exp(-\tfrac{1}{2}\eta_i A_{ij}^{-1} \eta_j)$$

where ω_j and η_j are real Grassmann variables and A_{ij} is a real, invertible antisymmetric matrix. [First establish the result for $\eta_j = 0$ by making an orthogonal transformation to a suitable block form for A.]

10. Evaluate directly

$$\delta(\omega_1, \ldots, \omega_n) = c \int d\eta_1 \ldots d\eta_n \, \exp i(\omega_1 \eta_1 + \ldots + \omega_n \eta_n)$$

where ω_j and η_j are real Grassmann variables and c is a constant. Hence show that $\delta(\omega_j)$ is indeed a delta-function for Grassmann integration:

$$\int d\omega_1 \ldots d\omega_n \, \delta(\omega_i) \, f(\omega_1, \ldots, \omega_n) = f(0, \ldots, 0)$$

for any function $f(\omega_j)$, provided c is a suitable complex number of unit modulus (which you need not determine).