Families of Hitchin systems and $\mathcal{N}$=2 theories

Aswin Balasubramanian

May 12, 2020

NHETC Seminar, Rutgers

Motivations

- Hitchin systems have appeared often in recent literature on extended SUSY theories. This has given us powerful geometric tools to study physics questions + non-trivial physics inspired tools to study the Hitchin system.
- One major source of such interaction has been the study of 4d $\mathcal{N} = 2$ Class S theories whose Coulomb branch geometry is encoded in the Hitchin system.
- When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT.

Motivations

Given this physics picture, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for $\mathfrak{j} = \mathrm{sl}_N$ + use dictionary to clarify behaviour of Coulomb branches.

Motivations

Given this physics picture, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family over $\overline{\mathcal{M}_{g,n}^{DM}}$.
Today's talk : Do this for $\mathfrak{j} = \mathrm{sl}_N$ + use dictionary to clarify behaviour of Coulomb branches.

Overview of the talk

1. $\mathcal{N} = 2$ theories and the Hitchin system
2. Hitchin system on a nodal curve (the local story)
3. Hitchin system over $\overline{\mathcal{M}}$ (the global story)

This talk is based on upcoming work with J. Distler and R. Donagi.

1. $\mathcal{N} = 2$ theories and the Hitchin system

$\mathcal{N} = 2$ *theories*

Seiberg and Witten came up with a strategy to solve for the low energy EFT at a generic point in the Coulomb branch B of any $\mathcal{N} = 2$ theory. They noted that this LE-EFT is encoded in a geometrical way that is constrained by IR electric-magnetic duality and extended SUSY.

$\mathcal{N} = 2$ *theories*

Seiberg and Witten came up with a strategy to solve for the low energy EFT at a generic point in the Coulomb branch B of any $\mathcal{N} = 2$ theory. They noted that this LE-EFT is encoded in a geometrical way that is constrained by IR electric-magnetic duality and extended SUSY.

SW integrable systems

The geometric data contains :

- In concise terms, the LE-EFT is encoded in a complex integrable system (Martinec-Warner, Donagi-Witten, Freed).
- The base B of a Seiberg-Witten integrable system carries a special-Kähler metric and is the Coulomb branch of the 4d $\mathcal{N}=2$ theory.
- The total space of the I.S can be identified with the Coulomb branch of the theory on $\mathbb{R}^{1,2}\times S^1_R$ at small values of R (Seiberg-Witten, Gaiotto-Moore-Neitzke).

SW integrable systems

The geometric data contains :

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition :
  $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

SW integrable systems

The geometric data contains :

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition : $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

The geometric data contains :

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \to \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$Z_{\Gamma_e} \equiv a = \int_{A-cycle} \lambda \quad (1)$$

$$Z_{\Gamma_m} \equiv a_D = \int_{B-cycle} \lambda \quad (2)$$

SW integrable systems

The geometric data contains :

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \rightarrow \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$Z_{\Gamma_e} \equiv a = \int_{A-cycle} \lambda \qquad (1)$$

$$Z_{\Gamma_m} \equiv a_D = \int_{B-cycle} \lambda \qquad (2)$$

Class $\mathcal{S}$ theories

- A large class of $\mathcal{N} = 2$ theories can be obtained by formulating the 6d (0,2) theory $\mathscr{X}(\mathfrak{j})$ ($\mathfrak{j} \in A, D, E$) on $\mathbb{R}^{1,3} \times C_{g,n}$ (with a partial twist) and dimensionally reducing on C. We also insert certain 4d 1/2 BPS defects of the 6d theory (or co-dimension two defects) at the n punctures. (Gaiotto, Gaiotto-Moore-Neitzke).
- The space of marginal parameters $\{\tau_i\}$ is identified with the moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$.

Class $\mathcal{S}$ theories

- A large class of $\mathcal{N} = 2$ theories can be obtained by formulating the 6d (0,2) theory $\mathscr{X}(\mathfrak{j})$ ($\mathfrak{j} \in A, D, E$) on $\mathbb{R}^{1,3} \times C_{g,n}$ (with a partial twist) and dimensionally reducing on C. We also insert certain 4d 1/2 BPS defects of the 6d theory (or co-dimension two defects) at the n punctures. (Gaiotto, Gaiotto-Moore-Neitzke).
- The space of marginal parameters $\{\tau_i\}$ is identified with the moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$.

Class $\mathcal{S}$ theories

- A large class of $\mathcal{N} = 2$ theories can be obtained by formulating the 6d (0,2) theory $\mathscr{X}(\mathfrak{j})$ ($\mathfrak{j} \in A, D, E$) on $\mathbb{R}^{1,3} \times C_{g,n}$ (with a partial twist) and dimensionally reducing on C. We also insert certain 4d 1/2 BPS defects of the 6d theory (or co-dimension two defects) at the n punctures. (Gaiotto, Gaiotto-Moore-Neitzke).
- The space of marginal parameters $\{\tau_i\}$ is identified with the moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$.

Class S theories and the Hitchin system

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V, ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V,ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

Class S theories and the Hitchin system

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V, ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

Class S theories and the Hitchin system

Consider the map

$$\mu : \mathcal{M}_H \to B,$$

where $B := \oplus_k H^0(C, K^{\otimes k})$, where k runs over the the degrees of invariant polynomials of $\mathfrak{j}$. We take $\mathfrak{j} = \mathfrak{sl}_N$.
Hitchin showed that μ is a Lagrangian fibration and the generic fibers of the map μ are Lagrangian tori.
The base of the Hitchin system parameterizes the Coulomb branch of the 4d Class S theory while the total space $\mathcal{M}_H$ parameterizes the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$.

Class S theories and the Hitchin system

Consider the map

$$\mu : \mathcal{M}_H \to B,$$

where $B := \oplus_k H^0(C, K^{\otimes k})$, where k runs over the the degrees of invariant polynomials of $\mathfrak{j}$. We take $\mathfrak{j} = \mathfrak{sl}_N$.
Hitchin showed that μ is a Lagrangian fibration and the generic fibers of the map μ are Lagrangian tori.
The base of the Hitchin system parameterizes the Coulomb branch of the 4d Class S theory while the total space $\mathcal{M}_H$ parameterizes the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$.

The SW curve is the spectral curve $\Sigma_b \equiv det_{\underline{N}}(\lambda I - \phi) = 0$ and the SW differential is $\lambda dz\,|_{\Sigma}$.

Now, we want to allow punctures. We restrict ourselves to :

- **Tame defects** : Defects where the Higgs one-form has a simple pole, $\phi = \frac{a}{z} dz + (\ldots)$, $a \in \mathfrak{j}$.

How does the integrable system look when we allow these punctures ?

The SW curve is the spectral curve $\Sigma_b \equiv det_{\underline{N}}(\lambda I - \phi) = 0$ and the SW differential is $\lambda dz \,|_{\Sigma}$.
Now, we want to allow punctures. We restrict ourselves to :

- **Tame defects** : Defects where the Higgs one-form has a simple pole, $\phi = \frac{a}{z} dz + (\dots)$, $a \in \mathfrak{j}$.

How does the integrable system look when we allow these punctures ?

Poisson vs Symplectic

With punctures : We again have the moduli space of pairs (V, ϕ) where and $\phi \in H^0(C, ad(V) \otimes K(D))$ and $B := \oplus_{k=2}^{N} H^0(C, K(D)^{\otimes k})$. This gives a Poisson integrable system.

This is because $\dim(B)$ is greater than $1/2 \dim(\mathcal{M}_H)$. The additional base parameters (sometimes called "Casimir parameters") correspond to the freedom to vary the residues $Res(\phi)$ at $E_i \in D$ in a fixed sheet of the Lie algebra $\mathfrak{j}$. In physics terms, they correspond to (local) mass deformations.

Poisson vs Symplectic

With punctures : We again have the moduli space of pairs (V, ϕ) where and $\phi \in H^0(C, ad(V) \otimes K(D))$ and $B := \oplus_{k=2}^{N} H^0(C, K(D)^{\otimes k})$. This gives a Poisson integrable system.

This is because $\dim(B)$ is greater than $1/2 \dim(\mathcal{M}_H)$. The additional base parameters (sometimes called "Casimir parameters") correspond to the freedom to vary the residues $Res(\phi)$ at $E_i \in D$ in a fixed sheet of the Lie algebra $\mathfrak{j}$. In physics terms, they correspond to (local) mass deformations.

Poisson vs Symplectic

We want the symplectic integrable system obtained by fixing $Res(\phi)_i$ to be in specific conjugacy classes. In fact, we want *all* of these residues to be nilpotent. Depending on the residues, B is modified :

$$B := \bigoplus_k H^0(C, \mathcal{L}_k) \tag{3}$$

where

$$\mathcal{L}_k = \Big(K_C(D)\Big)^{\otimes k} \otimes \mathcal{O}\Big(-\sum_{E_i \in D} \chi_k^i E_i\Big) \tag{4}$$

Henceforth, T.H.I.S is the symplectic I.S above.

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3).$

$\mathcal{T.H.I.S}$ on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

$\mathcal{T.H.I.S}$ on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

$\mathcal{T.H.I.S}$ on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Bad, Ugly and Good Hitchin systems

Motivated by the physics (Gaiotto-Razamat), we introduce the following definitions for a T.H.I.S :

- A *bad* Hitchin system : These have $h^1(C, \mathcal{L}_k) > 0$ for some k.
- An *ugly* Hitchin system : These are not bad but have a non-trivial kernel for $\kappa : \{m_i\}_{local} \to \{m_i\}_{global}$, the between the local and global Poisson deformation spaces.
- A *good* Hitchin system : These are neither bad nor ugly.

We mostly want to work with Hitchin systems that are "not bad" on smooth $C_{g,n}$. This turns out be a kind of stability condition (more on this later).

Bad, Ugly and Good Hitchin systems

Motivated by the physics (Gaiotto-Razamat), we introduce the following definitions for a T.H.I.S :

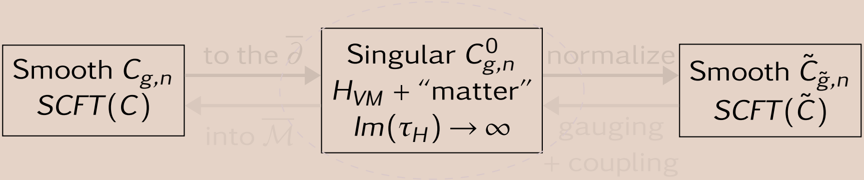
- A *bad* Hitchin system : These have $h^1(C, \mathcal{L}_k) > 0$ for some k.
- An *ugly* Hitchin system : These are not bad but have a non-trivial kernel for $\kappa : \{m_i\}_{local} \to \{m_i\}_{global}$, the between the local and global Poisson deformation spaces.
- A *good* Hitchin system : These are neither bad nor ugly.

We mostly want to work with Hitchin systems that are "not bad" on smooth $C_{g,n}$. This turns out be a kind of stability condition (more on this later).

2. Degeneration of T.H.I.S to a nodal curve

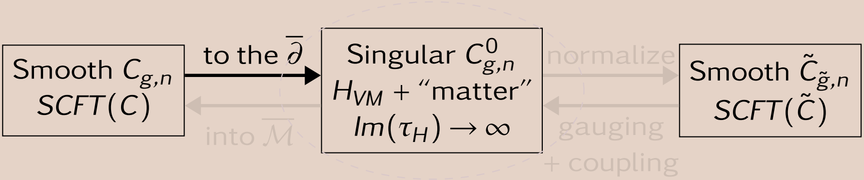
Going to the boundary of $\overline{\mathcal{M}}$

Physics at a co-dim 1 boundary in $\overline{\mathcal{M}^{DM}}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the $\overline{\partial}$. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

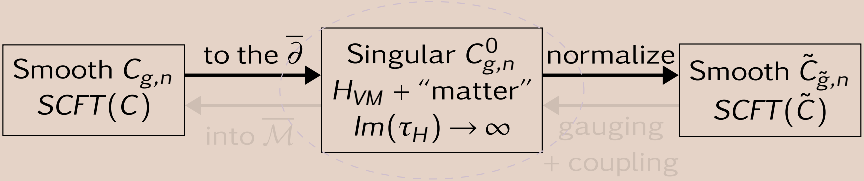
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}^{DM}}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the $\overline{\partial}$.
  Different H_i at different boundaries - a striking fact !
  (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

Going to the boundary of $\overline{\mathcal{M}}$

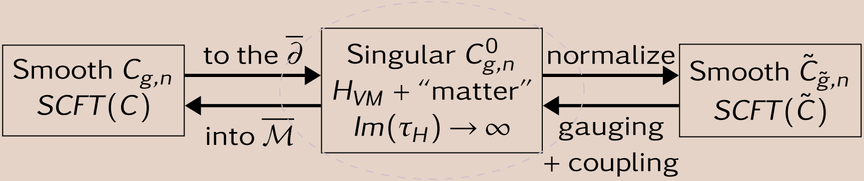
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}^{DM}}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the $\overline{\partial}$. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

Going to the boundary of $\overline{\mathcal{M}}$

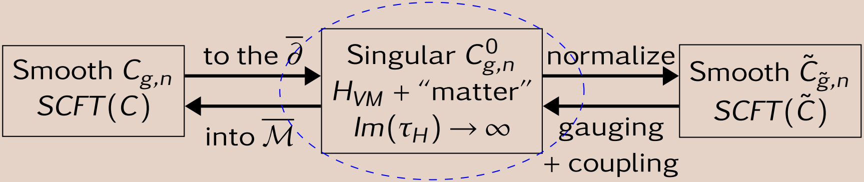
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}^{DM}}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the $\overline{\partial}$. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

Going to the boundary of $\overline{\mathcal{M}}$

Physics at a co-dim 1 boundary in $\overline{\mathcal{M}^{DM}}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the $\overline{\partial}$. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

The SCFT at a nodal UV curve

In the Class S context, this question has been studied and the rules that govern the relationship between $SCFT(C)$ and $SCFT(\widetilde{C})$ are known thanks to (Gaiotto, Chacaltana-Distler et al) and (Gaiotto-Moore-Tachikawa).
We want to *derive* these rules from a nodal Hitchin system.

Hitchin system on nodal curves

- The study of Bun_G on nodal curves has a long history. This can be thought of as a higher rank version of the study of *Jac* on a nodal curve.
- The study of *Higgs* bundles on nodal curves is more recent (Bhosle 2014, Balaji-Barik-Nagaraj 2016, Logares 2018). Specific examples on $\mathbb{P}^1$ had been studied from an integrable systems perspective (Nekrasov 1998, Chervov-Talalaev 2003).
- The goals in these above works were somewhat different from ours. In our paper, we develop the story from the basics.

Hitchin system on nodal curves

- The study of Bun_G on nodal curves has a long history. This can be thought of as a higher rank version of the study of Jac on a nodal curve.
- The study of *Higgs* bundles on nodal curves is more recent (Bhosle 2014, Balaji-Barik-Nagaraj 2016, Logares 2018). Specific examples on $\mathbb{P}^1$ had been studied from an integrable systems perspective (Nekrasov 1998, Chervov-Talalaev 2003).
- The goals in these above works were somewhat different from ours. In our paper, we develop the story from the basics.

Hitchin system on nodal curves

- We always work in complex structure I, ie the Higgs bundles picture.
- On a nodal curve C^0 with a node at p, there is a natural analog of the canonical bundle called the dualizing sheaf ω .
- This sheaf is defined as the pushforward of the sheaf of meromorphic differentials with at most simple poles at $\nu^{-1}(p)$ and subject to a residue condition. ($\nu : \tilde{C} \to C^0$ is the normalization).
- So, let us mimic the smooth story and carry everything over to the nodal case by replacing K with ω.

Non-Compact Fibers

An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system. The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters". One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Non-Compact Fibers

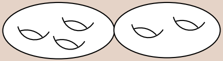
An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system. The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters". One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Non-Compact Fibers

An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system. The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters". One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Higgs bundles at a separating node

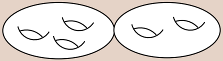
A separating node with $g_C = g_{C_L} + g_{C_R}$:



In this case, the Hitchin I.S factorizes into a left and a right integrable system which share some common compatibility data at the node p. Lets make this explicit.

Higgs bundles at a separating node

A separating node with $g_C = g_{C_L} + g_{C_R}$:



In this case, the Hitchin I.S factorizes into a left and a right integrable system which share some common compatibility data at the node p. Lets make this explicit.

Separating Node

Let us define certain sheaves :

$$\mathcal{L}_k\,|_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L} \tag{5}$$

$$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p) \tag{6}$$

$$\mathcal{L}_k\,|_{C_R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R} \tag{7}$$

$$\mathcal{L}_{k,R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R}(-p) \tag{8}$$

At the node, if we have a non-zero $h^0(\mathcal{L}_k|_{C_L})$ and $h^0(\mathcal{L}_k|_{C_R})$, there is a unique way to glue them together. This data is encoded in the value of a center parameter. One can think of this as the degree k piece of the image under μ of $Res(\phi)_{\{q,r\}}$ where $\{q,r\} = \nu^{-1}(p)$.

Separating Node

Let us define certain sheaves :

$$\mathcal{L}_k\,|_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L} \tag{5}$$

$$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p) \tag{6}$$

$$\mathcal{L}_k\,|_{C_R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R} \tag{7}$$

$$\mathcal{L}_{k,R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R}(-p) \tag{8}$$

At the node, if we have a non-zero $h^0(\mathcal{L}_k|_{C_L})$ and $h^0(\mathcal{L}_k|_{C_R})$, there is a unique way to glue them together. This data is encoded in the value of a center parameter. One can think of this as the degree k piece of the image under μ of $Res(\phi)_{\{q,r\}}$ where $\{q,r\} = \nu^{-1}(p)$.

Separating Node

Let b_k^C be the graded dimension of the space of center parameters. By construction, $b_k^C = 0, 1$ for each k.
Let us also define :

$$B_L := H^0(C, \mathcal{L}_{k,L}) \tag{9}$$

$$B_R := H^0(C, \mathcal{L}_{k,R}) \tag{10}$$

These vector spaces will end up being the bases of the left/right integrable system. Let b_k^L, b_k^R be the corresponding graded dimensions. What are the allowed separating nodes ?

Separating Node

Let b_k^C be the graded dimension of the space of center parameters. By construction, $b_k^C = 0, 1$ for each k.
Let us also define :

$$B_L := H^0(C, \mathcal{L}_{k,L}) \tag{9}$$

$$B_R := H^0(C, \mathcal{L}_{k,R}) \tag{10}$$

These vector spaces will end up being the bases of the left/right integrable system. Let b_k^L, b_k^R be the corresponding graded dimensions. What are the allowed separating nodes ?

Types of Nodes

$$\mathcal{L}_k|_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}$$
$$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p)$$

There are 3 possibilities for a T.H.I.S that is not bad :

1. **Standard Node** : This is a node in which $h^1(\mathcal{L}_{k,L}), h^1(\mathcal{L}_{k,R}) = 0$ and $b_k^C = 1$ for all k
2. **Regular Restricted Node** : This is a node at which $h^1(\mathcal{L}_{k,L}) > 0$ and/or $h^1(\mathcal{L}_{k,R}) > 0$ for some k and $b_k^C = 0$ for such k.
3. **Non-regular Restricted Node** : This is a node at which $h^1(\mathcal{L}_k|_{C_L}) > 0$ or $h^1(\mathcal{L}_k|_{C_R}) > 0$ for some k and $b_k^C = 0$ for such k. There is a new phenomenon here : In the nodal limit, $h^0(\mathcal{L}_k)$ and $h^1(\mathcal{L}_k)$ increase while keeping $h^0 - h^1$ constant.

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}_{\,0} + \cancel{b_k^C}_{\,0} + b_k^R > b_k^{smooth} \tag{11}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix. Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{12}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}_0 + \cancel{b_k^C}_0 + b_k^R > b_k^{smooth} \tag{11}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.

Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{12}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S … what are the allowed $(\vec{\chi}, \vec{b^C})$?

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}\,0 + \cancel{b_k^C}\,0 + b_k^R > b_k^{smooth} \tag{11}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.
Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{12}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}\,0 + \cancel{b_k^C}\,0 + b_k^R > b_k^{smooth} \tag{11}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.
Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{12}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

Theorem

Theorem In type A tame Hitchin systems that are not bad :

- A The vanishing orders χ_k^{node} uniquely determine a nilpotent orbit $\mathcal{O}$.
- B The resulting values of b_k^C are such that the space of center parameters can always be interpreted as the invariant polynomials for some $H \subset SU(N)$, where $H = SU(k), Sp(k)$ for some k.
- C If $[p_i]$ is the partition corresponding to $\mathcal{O}$, then we always have $p_1 - 2 \leq 2\text{rank}(H) \leq 2(p_1 - p_2 - 1)$.

The theorem associates a pair $(\mathcal{O}, H)$ to any node. In this notation, $([N], SU(N))$ is the standard node. Non-separating nodes are always standard.

Theorem

Theorem In type A tame Hitchin systems that are not bad :

- **A** The vanishing orders χ_k^{node} uniquely determine a nilpotent orbit $\mathcal{O}$.
- **B** The resulting values of b_k^C are such that the space of center parameters can always be interpreted as the invariant polynomials for some $H \subset SU(N)$, where $H = SU(k), Sp(k)$ for some k.
- **C** If $[p_i]$ is the partition corresponding to $\mathcal{O}$, then we always have $p_1 - 2 \leq 2\text{rank}(H) \leq 2(p_1 - p_2 - 1)$.

The theorem associates a pair $(\mathcal{O}, H)$ to any node. In this notation, $([N], SU(N))$ is the standard node. Non-separating nodes are always standard.

Consequences

Each part was expected from the physics picture but we prove them mathematically using our definition of nodal Hitchin system. This is reassuring. Also, our part (C) is a bit stronger as an *a priori* result than what was already known (unitarity & $\beta_{g_H^2} = 0$ constrain flavor center charges k_L, k_R and hence the allowed H for any fixed O).

Geometrically, one should think of the theorem as constraining the kind of singular spectral curves Σ_b that arise in the nodal limit. There are also similarities between these geometries and the kind of geometries arising when mathematicians study (geometric) twisted endoscopy using the Hitchin fibration.

Consequences

Each part was expected from the physics picture but we prove them mathematically using our definition of nodal Hitchin system. This is reassuring. Also, our part (C) is a bit stronger as an *a priori* result than what was already known (unitarity & $\beta_{g_H^2} = 0$ constrain flavor center charges k_L, k_R and hence the allowed H for any fixed O).

Geometrically, one should think of the theorem as constraining the kind of singular spectral curves Σ_b that arise in the nodal limit. There are also similarities between these geometries and the kind of geometries arising when mathematicians study (geometric) twisted endoscopy using the Hitchin fibration.

Relation to Deligne-Simpson

More about the restricted nodes :
At *every* restricted node, $h^1(\mathcal{L}_{k,L}) > 0$ and/or $h^1(\mathcal{L}_{k,R}) > 0$.
$\Longrightarrow$ We have a 'bad' Hitchin system on C_L and/or C_R. What does this mean ?
This happens only for $C_{L,R} = C_{0,p}$ for some p.
One can then translate this 'badness' to a condition on a corresponding character variety which is the moduli of irreps $\rho : \pi_1(C_{0,p+1}) \to J_{\mathbb{C}}$ where we fix a regular conjugacy class at the pre-image of node (the $(p+1)-th$ point).
Existence/Non-existence of such irreps is related to the existence/non-existence of semi-stable (quasi-) parabolic Higgs bundles

Relation to Deligne-Simpson

More about the restricted nodes :
At *every* restricted node, $h^1(\mathcal{L}_{k,L}) > 0$ and/or $h^1(\mathcal{L}_{k,R}) > 0$.
$\Longrightarrow$ We have a 'bad' Hitchin system on C_L and/or C_R. What does this mean ?
This happens only for $C_{L,R} = C_{0,p}$ for some p.
One can then translate this 'badness' to a condition on a corresponding character variety which is the moduli of irreps $\rho : \pi_1(C_{0,p+1}) \rightarrow J_{\mathbb{C}}$ where we fix a regular conjugacy class at the pre-image of node (the $(p+1) - th$ point).
Existence/Non-existence of such irreps is related to the existence/non-existence of semi-stable (quasi-) parabolic Higgs bundles

Relation to Deligne-Simpson

Studying the existence of such irreps is called the Deligne-Simpson problem. The problem asks under what conditions on conjugacy classes (in SL_N) $C_1, C_2, \ldots C_{p+1}$ can we find matrices $M_i \in C_i$ that obey $M_1 M_2 \ldots M_{p+1} = \mathrm{Id}$.

When one of the conjugacy classes, say C_{p+1}, is regular, the problem was solved by Simpson.

A consequence of our theorem : At every restricted node, we have a Hitchin system (on C_L and/or C_R) for which the corresponding Deligne-Simpson problem with a regular C_{p+1} does not have a solution.

In other words, bad/non-bad $\Leftrightarrow$ unstable/semi-stable Higgs bundle (in these cases).

Relation to Deligne-Simpson

Studying the existence of such irreps is called the Deligne-Simpson problem. The problem asks under what conditions on conjugacy classes (in SL_N) $C_1, C_2, \ldots C_{p+1}$ can we find matrices $M_i \in C_i$ that obey $M_1 M_2 \ldots M_{p+1} = \mathrm{Id}$.
When one of the conjugacy classes, say C_{p+1}, is regular, the problem was solved by Simpson.

A consequence of our theorem : At every restricted node, we have a Hitchin system (on C_L and/or C_R) for which the corresponding Deligne-Simpson problem with a regular C_{p+1} does not have a solution.
In other words, bad/non-bad $\Leftrightarrow$ unstable/semi-stable Higgs bundle (in these cases).

Relation to Deligne-Simpson

Studying the existence of such irreps is called the Deligne-Simpson problem. The problem asks under what conditions on conjugacy classes (in SL_N) $C_1, C_2, \dots C_{p+1}$ can we find matrices $M_i \in C_i$ that obey $M_1 M_2 \dots M_{p+1} = \mathrm{Id}$.
When one of the conjugacy classes, say C_{p+1}, is regular, the problem was solved by Simpson.
A consequence of our theorem : At every restricted node, we have a Hitchin system (on C_L and/or C_R) for which the corresponding Deligne-Simpson problem with a regular C_{p+1} does not have a solution.
In other words, bad/non-bad $\Leftrightarrow$ unstable/semi-stable Higgs bundle (in these cases).

Relation to Deligne-Simpson

This appearance of unstable Higgs bundles on $C_{L/R}$ implies that the structure group of the underlying principal bundle is reduced.

This property of the **Coulomb branch** should be thought of as analogous to a property that the **Higgs branch** is conjectured to obey (Gaiotto-Moore-Tachikawa). The GMT conjecture says that the smaller groups H should be interpreted as the (hyper-Kähler) isometries of 4d Higgs branches of bad theories $HB(C_{L/R})$.

H. Nakajima has studied the GMT conjecture for the Higgs branch of some of the bad theories and his results are compatible with ours.

3. Global Story

Global Story for $\overline{\mathcal{M}}_{0,4}$

We consider the following global model for the **universal curve** $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{0,4}$.
Consider $\mathbb{CP}^2$ blown up at four points :
$E_1 \to (1,0,0)$, $E_2 \to (0,1,0)$, $E_3 \to (0,0,1)$, $E_4 \to (1,1,1)$. Let us denote the blown up surface as $\widetilde{\mathbb{CP}}^2$. Let λ_1, λ_2 be homogeneous co-ordinates on $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$. (cross ratio $\lambda = \lambda_1/\lambda_2$).
We identify the universal curve, $\mathcal{C} \simeq \widetilde{\mathbb{CP}}^2$ and the projection $\pi : \widetilde{\mathbb{CP}}^2 \to \overline{\mathcal{M}}_{0,4}$ is defined as the solution to

$$\lambda_1 x(y-z) + \lambda_2 y(z-x) = 0 \qquad (13)$$

Global Story for $\overline{\mathcal{M}}_{0,4}$

We have

$$\mathcal{L}_k = \mathcal{O}(k)\Big(-\sum_i \chi_k^i E_i\Big)$$

On each curve, C_λ, the ϕ_k are holomorphic sections of $\mathcal{L}_k|_{C_\lambda}$. These fit together to form

$$\phi_k \in H^0(\mathcal{C}, \mathcal{L}_k) = H^0(\overline{\mathcal{M}}_{0,4}, \pi_* \mathcal{L}_k)$$

We want to compute the direct image sheaves, $\pi_* \mathcal{L}_k$ on $\overline{\mathcal{M}}_{0,4}$.

Global Story for $\overline{\mathcal{M}_{0,4}}$

But any vector bundle on $\mathbb{CP}^1$ splits as a direct sum of line bundles. So,

$$\pi_* \mathcal{O}(k)\Big(-\sum_i \chi_k^i E_i\Big) = \sum_{i \in \mathbb{Z}} m_i \mathcal{O}_{\mathbb{P}^1}(i) \tag{14}$$

for some $m_i \geq 0$.
We actually compute m_i for any $Res(\phi)_i$ at E_i. The result is quite surprising. Set $\chi_k^i = 1$. In this case, the answer is :

- $\mathbf{k = 2p}$: m_l is non-zero only for $l = 1, \dots p$ and looks like $m_l = (4, 4, 4, \dots, 1)$
- $\mathbf{k = 2p + 1}$: m_l is non-zero only for $l = 1, \dots p$ and looks like $m_l = (4, 4, 4, \dots, 3)$

But any vector bundle on $\mathbb{CP}^1$ splits as a direct sum of line bundles. So,

$$\pi_*\mathcal{O}(k)\Big(-\sum_i \chi_k^i E_i\Big) = \sum_{i\in\mathbb{Z}} m_i \mathcal{O}_{\mathbb{P}^1}(i) \quad (14)$$

for some $m_i \geq 0$.
We actually compute m_i for any $Res(\phi)_i$ at E_i. The result is quite surprising. Set $\chi_k^i = 1$. In this case, the answer is :

- $\mathbf{k = 2p}$: m_l is non-zero only for $l = 1, \dots p$ and looks like $m_l = (4, 4, 4, \dots, 1)$
- $\mathbf{k = 2p + 1}$: m_l is non-zero only for $l = 1, \dots p$ and looks like $m_l = (4, 4, 4, \dots, 3)$

Global Story for $\overline{\mathcal{M}_{0,4}}$

But any vector bundle on $\mathbb{CP}^1$ splits as a direct sum of line bundles. So,

$$\pi_*\mathcal{O}(k)\Big(-\sum_i \chi_k^i E_i\Big) = \sum_{i\in\mathbb{Z}} m_i \mathcal{O}_{\mathbb{P}^1}(i) \qquad (14)$$

for some $m_i \geq 0$.

We actually compute m_i for any $Res(\phi)_i$ at E_i. The result is quite surprising. Set $\chi_k^i = 1$. In this case, the answer is :

- $\mathbf{k = 2p}$: m_l is non-zero only for $l = 1, \ldots p$ and looks like $m_l = (4, 4, 4, \ldots, 1)$
- $\mathbf{k = 2p + 1}$: m_l is non-zero only for $l = 1, \ldots p$ and looks like $m_l = (4, 4, 4, \ldots, 3)$

Global Story for $\overline{\mathcal{M}_{0,4}}$

Ex in SL_4 : $([4],[4],[4],[4])$. This has a 9 dim base with $b_k = (1,3,5)$. As bundles on $\overline{M_{0,4}} \simeq \mathbb{CP}^1$, B_k split as

$$B_2 = \mathcal{O}(1) \tag{15}$$
$$B_3 = 3\mathcal{O}(1) \tag{16}$$
$$B_4 = 4\mathcal{O}(1) + \mathcal{O}(2). \tag{17}$$

We believe that m_i constitute new information about Class $S[C_{0,4}]$ theories. We expect them to be useful in calculating various observables of the theory that have a non-trivial dependence on $\lambda \in \mathcal{M}_{0,4}$. For ex : The u-plane integrals arising in topologically twisted theories, Z_Ω etc.
One can also use the global model to study types of nodes, flatness automatic!

Further Applications

- We now have a dictionary between the nodal Hitchin system and the physics of $\mathcal{N} = 2$ theories. One can now imagine using a lot of other Class S tools (like GMN spectral networks) to probe the nodal Hitchin geometry further.
- One could use our results + non-abelian Hodge theory to study geometry of the character variety $\mathcal{M}_{B(etti)} \sim$ Mod. Sp.$(\rho : \pi_1(C_{g,n}) \to J_{\mathbb{C}})$. Topology of $\mathcal{M}_B$, General Deligne-Simpson problem, Higher Fenchel-Nielsen co-ordinates.
- Extend all of this to other Cartan types $\mathfrak{j} \in D, E$.

Example

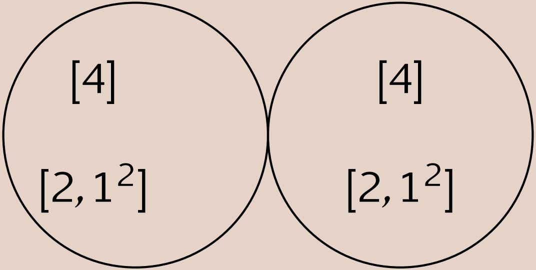
Consider a SL_4 Hitchin system on $C_{0.4}$ with residues in $([4],[4],[2,1^2],[2,1^2])$.
The orders of the zeroes of the ϕ_k are:

E_i	O_H	χ_2	χ_3	χ_4
E_1	$[2,1^2]$	1	2	3
E_2	$[2,1^2]$	1	2	3
E_3	$[4]$	1	1	1
E_4	$[4]$	1	1	1

$\overline{\mathcal{M}_{0,4}}$ has three boundary points.

Example

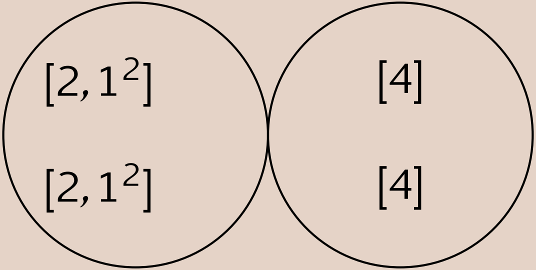
Near two of them, we have a degeneration of the type : (limits correspond to to $SU(4), N_f = 8$ theories)



They are both **standard**, ie we would assign the pair $(O, H) = ([N], SU(N))$ to the nodes.

Example

The other node is of the type : (corresponding to a $SU(2)$ gauge theory coupled to a R_4 SCFT + hypers in $\underline{2}$)



This is an example of a **non-regular restricted node.** We assign $(O,H) = ([3,1], SU(2))$ to this node.