Families of Hitchin systems, $\mathcal{N}$=2 theories and the DS problem

Aswin Balasubramanian

Rutgers University
ICTS-TIFR

Jan 27, 2021

Strings and QFT Seminar, Fudan University

Motivations

- Hitchin systems have appeared often in recent literature on extended SUSY theories.
- This has given us powerful geometric tools to study physics questions + non-trivial physics inspired tools to study the Hitchin system.
- One major source of such interaction has been the study of 4d $\mathcal{N} = 2$ Class S theories whose Coulomb branch geometry is encoded in the Hitchin system.
- Extremely active interface between physics & math : spectral networks, the AGT conjecture and its extensions, Nekrasov-Shatashivili, Geometric Langlands, ...
- The $\mathcal{N} = 2 \leftrightarrow$ Hitchin systems dictionary plays an important role in each of these.

Motivations

When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT. Non-perturbative dualities of these theories have a geometric interpretation in terms of moving to different boundaries in $\overline{\mathcal{M}_C}^{DM}$.

Given this, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family of integrable systems over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for (tame) Hitchin system of type A + explain some surprises we found along the way.

Motivations

When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT. Non-perturbative dualities of these theories have a geometric interpretation in terms of moving to different boundaries in $\overline{\mathcal{M}_C}^{DM}$.

Given this, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family of integrable systems over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for (tame) Hitchin system of type A + explain some surprises we found along the way.

Motivations

When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT. Non-perturbative dualities of these theories have a geometric interpretation in terms of moving to different boundaries in $\overline{\mathcal{M}_C}^{DM}$.

Given this, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family of integrable systems over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for (tame) Hitchin system of type A + explain some surprises we found along the way.

Motivations

When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}^{DM}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT. Non-perturbative dualities of these theories have a geometric interpretation in terms of moving to different boundaries in $\overline{\mathcal{M}}_C^{DM}$.

Given this, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family of integrable systems over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for (tame) Hitchin system of type A + explain some surprises we found along the way.

Motivations

When the relevant $\mathcal{N} = 2$ theory is a SCFT, the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$ associated to the UV Curve is identified with the space of marginal parameters of the SCFT. Non-perturbative dualities of these theories have a geometric interpretation in terms of moving to different boundaries in $\overline{\mathcal{M}_C}^{DM}$.

Given this, it is very natural to ask what becomes of the Hitchin system when we vary $\tau \in \overline{\mathcal{M}^{DM}}$?

Broad goal of our project : Build a dictionary between physics and the behaviour of the Hitchin system as a family of integrable systems over $\overline{\mathcal{M}_{g,n}^{DM}}$.

Today's talk : Do this for (tame) Hitchin system of type A + explain some surprises we found along the way.

Overview of the talk

1. $\mathcal{N}=2$ theories and the Hitchin system
2. Hitchin system on a nodal curve
3. Connection to the Deligne-Simpson problem

This talk is based on 2008.01020 with J. Distler and R. Donagi + ongoing work.

1. $\mathcal{N} = 2$ theories and the Hitchin system

$\mathcal{N} = 2$ theories

- Seiberg and Witten came up with a strategy to solve for the low energy EFT at a generic point in the Coulomb branch B of a 4d $\mathcal{N} = 2$ theory.
- They noted that this LE-EFT is encoded in a geometrical way that is constrained by IR electric-magnetic duality and extended SUSY.

- In concise terms, the LE-EFT is encoded in a complex integrable system (Martinec-Warner, Donagi-Witten, Freed).
- The base B of a Seiberg-Witten integrable system carries a special-Kähler metric and is the Coulomb branch of the 4d $\mathcal{N} = 2$ theory.
- The total space of the I.S can be identified with the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$ at small values of R (Seiberg-Witten, Gaiotto-Moore-Neitzke).

$\mathcal{N} = 2$ theories

- Seiberg and Witten came up with a strategy to solve for the low energy EFT at a generic point in the Coulomb branch B of a 4d $\mathcal{N} = 2$ theory.
- They noted that this LE-EFT is encoded in a geometrical way that is constrained by IR electric-magnetic duality and extended SUSY.
- In concise terms, the LE-EFT is encoded in a complex integrable system (Martinec-Warner, Donagi-Witten, Freed).
- The base B of a Seiberg-Witten integrable system carries a special-Kähler metric and is the Coulomb branch of the 4d $\mathcal{N} = 2$ theory.
- The total space of the I.S can be identified with the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$ at small values of R (Seiberg-Witten, Gaiotto-Moore-Neitzke).

SW integrable systems

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition : $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \to \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$
\begin{aligned}
Z_{\Gamma_e} &\equiv a = \int_{A-cycle} \lambda \\
Z_{\Gamma_m} &\equiv a_D = \int_{B-cycle} \lambda
\end{aligned}
$$

SW integrable systems

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition : $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \to \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$Z_{\Gamma_e} \equiv a = \int_{A-cycle} \lambda$$

$$Z_{\Gamma_m} \equiv a_D = \int_{B-cycle} \lambda$$

SW integrable systems

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition : $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \to \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$
\begin{aligned}
Z_{\Gamma_e} &\equiv a = \int_{A-cycle} \lambda \\
Z_{\Gamma_m} &\equiv a_D = \int_{B-cycle} \lambda
\end{aligned}
$$

SW integrable systems

- Typically, the fibers F_b of SW integrable systems turn out to be $Jac(\Sigma_b), Prym(\Sigma_b)$, where Σ is the SW curve.
- The lattice of EM charges Γ is identified with $H_1(\Sigma_b, \mathbb{Z})$
- There is a symplectic pairing on Γ that arises from the Dirac-Schwinger-Zwanziger condition : $(e_1 m_2 - e_2 m_1) = 2\pi\mathbb{Z}$ for dyons.
- The choice of a splitting $\Gamma = \Gamma_e \oplus \Gamma_m$ gives a choice of a principal polarization for F_b where τ_{ij}^{IR} is the period matrix.

- And the $\mathcal{N} = 2$ central charge function $Z : \Gamma \to \mathbb{C}$ is given in terms of period integrals of a meromorphic one-form λ called the SW one-form :

$$Z_{\Gamma_e} \equiv a = \int_{A-cycle} \lambda$$

$$Z_{\Gamma_m} \equiv a_D = \int_{B-cycle} \lambda$$

Class $\mathcal{S}$ theories

- A large class of $\mathcal{N}=2$ theories can be obtained by formulating the 6d (0,2) theory $\mathscr{X}(\mathfrak{j})$ ($\mathfrak{j} \in A, D, E$) on $\mathbb{R}^{1,3} \times C_{g,n}$ (with a partial twist) and dimensionally reducing on C. We also insert certain 4d 1/2 BPS defects of the 6d theory (or co-dimension two defects) at the n punctures. (Gaiotto, Gaiotto-Moore-Neitzke).
- The space of marginal parameters $\{\tau_i\}$ is identified with the moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$.

Class S theories

- A large class of $\mathcal{N} = 2$ theories can be obtained by formulating the 6d (0,2) theory $\mathscr{X}(\mathfrak{j})$ ($\mathfrak{j} \in A, D, E$) on $\mathbb{R}^{1,3} \times C_{g,n}$ (with a partial twist) and dimensionally reducing on C. We also insert certain 4d 1/2 BPS defects of the 6d theory (or co-dimension two defects) at the n punctures. (Gaiotto, Gaiotto-Moore-Neitzke).
- The space of marginal parameters $\{\tau_i\}$ is identified with the moduli space $\overline{\mathcal{M}_{g,n}^{DM}}$.

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V, ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V, ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

Class S theories and the Hitchin system

The Seiberg-Witten I.S. associated to a Class S theory is a Hitchin system on $C_{g,n}$ of type $\mathfrak{j}$.

Without punctures :

The Hitchin system is a complex integrable system whose total space $\mathcal{M}_H$ is the moduli space of pairs (V, ϕ) where V is a principal $\mathfrak{j}$-bundle and $\phi \in H^0(C, ad(V) \otimes K)$. (+ suitable stability condition).

Consider the map

$$\mu : \mathcal{M}_H \to B,$$

where $B := \oplus_k H^0(C, K^{\otimes k})$, where k runs over the the degrees of invariant polynomials of $\mathfrak{j}$. We take $\mathfrak{j} = \mathfrak{sl}_N$.
Hitchin showed that μ is a Lagrangian fibration and the generic fibers of the map μ are Lagrangian tori.
The base of the Hitchin system parameterizes the Coulomb branch of the 4d Class S theory while the total space $\mathcal{M}_H$ parameterizes the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$.

Consider the map

$$\mu : \mathcal{M}_H \to B,$$

where $B := \oplus_k H^0(C, K^{\otimes k})$, where k runs over the the degrees of invariant polynomials of $\mathfrak{j}$. We take $\mathfrak{j} = \mathfrak{sl}_N$.

Hitchin showed that μ is a Lagrangian fibration and the generic fibers of the map μ are Lagrangian tori.

The base of the Hitchin system parameterizes the Coulomb branch of the 4d Class S theory while the total space $\mathcal{M}_H$ parameterizes the Coulomb branch of the theory on $\mathbb{R}^{1,2} \times S^1_R$.

Class S theories and the Hitchin system

The SW curve is the spectral curve $\Sigma_b \equiv det_{\underline{N}}(\lambda I - \phi) = 0$ and the SW differential is $\lambda dz \,|_{\Sigma}$.

Now, we want to allow punctures. We restrict ourselves to :

- **Tame defects** : Defects where the Higgs one-form has a simple pole, $\phi = \frac{a}{z} dz + (\dots)$, $a \in \mathfrak{j}$.

How does the integrable system look when we allow these punctures ?

The SW curve is the spectral curve $\Sigma_b \equiv det_{\underline{N}}(\lambda I - \phi) = 0$ and the SW differential is $\lambda dz\,|_{\Sigma}$.

Now, we want to allow punctures. We restrict ourselves to :

- **Tame defects** : Defects where the Higgs one-form has a simple pole, $\phi = \frac{a}{z} dz + (\dots)$, $a \in \mathfrak{j}$.

How does the integrable system look when we allow these punctures ?

With punctures : We again have the moduli space of pairs (V, ϕ) where and $\phi \in H^0(C, ad(V) \otimes K(D))$ and $B := \oplus_{k=2}^{N} H^0(C, K(D)^{\otimes k})$. This gives a Poisson integrable system.

This is because $\dim(B)$ is greater than $1/2 \dim(\mathcal{M}_H)$. The additional base parameters (sometimes called "Casimir parameters") correspond to the freedom to vary the residues $Res(\phi)$ at $E_i \in D$ in a fixed sheet of the Lie algebra $\mathfrak{j}$. In physics terms, they correspond to (local) mass deformations (see my earlier work with J. Distler).

Poisson vs Symplectic

With punctures : We again have the moduli space of pairs (V, ϕ) where and $\phi \in H^0(C, ad(V) \otimes K(D))$ and $B := \oplus_{k=2}^{N} H^0(C, K(D)^{\otimes k})$. This gives a Poisson integrable system.

This is because $\dim(B)$ is greater than $1/2 \dim(\mathcal{M}_H)$. The additional base parameters (sometimes called "Casimir parameters") correspond to the freedom to vary the residues $Res(\phi)$ at $E_i \in D$ in a fixed sheet of the Lie algebra $\mathfrak{j}$. In physics terms, they correspond to (local) mass deformations (see my earlier work with J. Distler).

Poisson vs Symplectic

We want the symplectic integrable system obtained by fixing $Res(\phi)_i$ to be in specific conjugacy classes. In fact, we want *all* of these residues to be nilpotent. Depending on the residues, B is modified :

$$B := \bigoplus_k H^0(C, \mathcal{L}_k) \tag{1}$$

where

$$\mathcal{L}_k = \Big(K_C(D)\Big)^{\otimes k} \otimes \mathcal{O}\Big(-\sum_{E_i \in D} \chi_k^i E_i\Big) \tag{2}$$

Henceforth, T.H.I.S is the symplectic I.S above.

$\mathcal{T}.\mathcal{H}.\mathcal{I}.\mathcal{S}$ on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4,3,1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1,1,1,2,2,2,3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1,\ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4,3,1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1,1,1,2,2,2,3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1,\ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1, \ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1, \ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1'̓s, the 2nd column with '2''s and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1, \ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \to (1, \ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{YT\} \rightarrow (1, \ldots)$?

T.H.I.S on smooth $C_{g,n}$

Simple algorithm for χ_k. Let $[p_i]$ be the partition label corresponding to the nilpotent conjugacy class. Represent this as a YD by using its parts as column sizes. Now, just fill the 1st column with '1's, the 2nd column with '2's and so on. Write the numbers in the YD column-wise, dropping the leading '1' to get χ_k. For ex, let $[p_i] = [4, 3, 1]$ in SL_8. We represent it by the following Young diagram:

4	3	1
1	2	3
1	2	
1	2	
1		

$\chi_k = (1, 1, 1, 2, 2, 2, 3)$.

Exercise : Which sequences are in the image of the map $\vec{\chi} : \{\text{YT}\} \to (1, \ldots)$?

OK/Bad Hitchin systems

Motivated by the physics (Gaiotto-Razamat), we introduce the following definitions for a T.H.I.S :

- A *bad* Hitchin system : These have $h^1(C, \mathcal{L}_k) > 0$ for some k.
- An *ugly* Hitchin system : These are not bad but have a non-trivial kernel for $\kappa : \{m_i\}_{local} \to \{m_i\}_{global}$, the between the local and global Poisson deformation spaces. Presence of *free hypers*.
- A *good* Hitchin system : These are neither bad nor ugly.

We mostly want to work with Hitchin systems that are "not bad" on smooth $C_{g,n}$. So, we gave them a simpler name : *OK* Hitchin systems. This turns out be a kind of stability condition (more on this later).

OK/Bad Hitchin systems

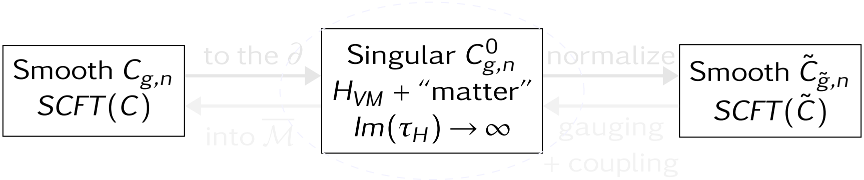
Motivated by the physics (Gaiotto-Razamat), we introduce the following definitions for a T.H.I.S :

- A *bad* Hitchin system : These have $h^1(C, \mathcal{L}_k) > 0$ for some k.
- An *ugly* Hitchin system : These are not bad but have a non-trivial kernel for $\kappa : \{m_i\}_{local} \to \{m_i\}_{global}$, the between the local and global Poisson deformation spaces. Presence of *free hypers*.
- A *good* Hitchin system : These are neither bad nor ugly.

We mostly want to work with Hitchin systems that are "not bad" on smooth $C_{g,n}$. So, we gave them a simpler name : *OK* Hitchin systems. This turns out be a kind of stability condition (more on this later).

2. Degeneration of T.H.I.S to a nodal curve

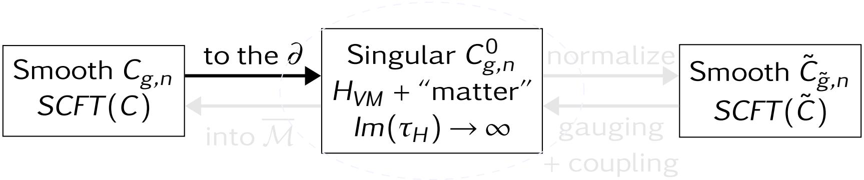
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}}^{DM}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the ∂. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

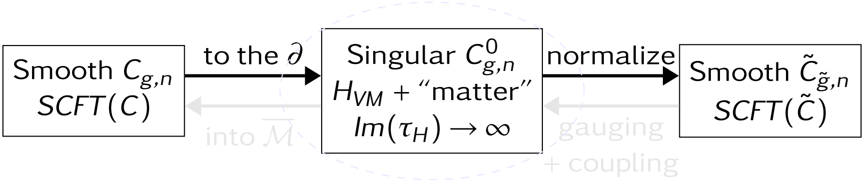
Going to the boundary of $\overline{\mathcal{M}}$

Physics at a co-dim 1 boundary in $\overline{\mathcal{M}}^{DM}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the ∂. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

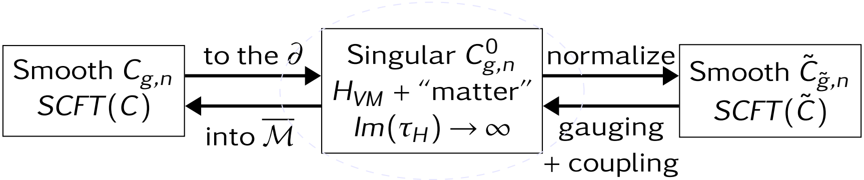
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}}^{DM}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the ∂. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

Going to the boundary of $\overline{\mathcal{M}}$

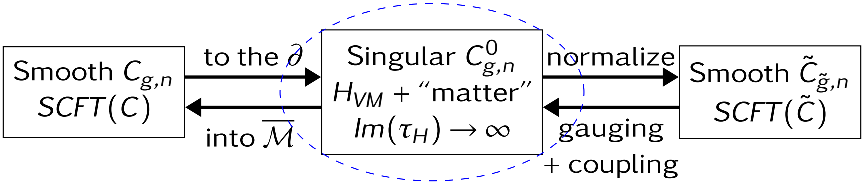
Physics at a co-dim 1 boundary in $\overline{\mathcal{M}}^{DM}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the ∂. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

Going to the boundary of $\overline{\mathcal{M}}$

Physics at a co-dim 1 boundary in $\overline{\mathcal{M}}^{DM}$:



- $H \subset SU(N)$ is a weakly coupled gauge group near the ∂. Different H_i at different boundaries - a striking fact ! (generalized Argyres-Seiberg duality).
- The theory on the nodal curve $C^0_{g,n}$ mediates between the two theories on smooth curves C and $\widetilde{C}$.

The SCFT at a nodal UV curve

In the Class S context, this question has been studied and the rules that govern the relationship between $SCFT(C)$ and $SCFT(\widetilde{C})$ are known thanks to (Gaiotto, Chacaltana-Distler et al) and (Gaiotto-Moore-Tachikawa).
We want to *derive* these rules from a nodal Hitchin system.

Hitchin system on nodal curves

- The study of Bun_G on nodal curves has a long history. This can be thought of as a higher rank version of the study of *Jac* on a nodal curve.
- The study of *Higgs* bundles on nodal curves is more recent (Bhosle 2014, Balaji-Barik-Nagaraj 2016, Logares 2018). Specific examples on $\mathbb{P}^1$ had been studied from an integrable systems perspective (Nekrasov 1998, Chervov-Talalaev 2003).
- The goals in these above works were somewhat different from ours. In our paper, we develop the story from the basics.

Hitchin system on nodal curves

- The study of Bun_G on nodal curves has a long history. This can be thought of as a higher rank version of the study of *Jac* on a nodal curve.
- The study of *Higgs* bundles on nodal curves is more recent (Bhosle [2014], Balaji-Barik-Nagaraj [2016], Logares [2018]). Specific examples on $\mathbb{P}^1$ had been studied from an integrable systems perspective (Nekrasov [1998], Chervov-Talalaev [2003]).
- The goals in these above works were somewhat different from ours. In our paper, we develop the story from the basics.

Hitchin system on nodal curves

- We always work in complex structure I, ie the Higgs bundles picture.
- On a nodal curve C^0 with a node at p, there is a natural analog of the canonical bundle called the dualizing sheaf ω .
- This sheaf is defined as the pushforward of the sheaf of meromorphic differentials with at most simple poles at $\nu^{-1}(p)$ and subject to a residue condition. ($\nu : \tilde{C} \to C^0$ is the normalization).
- So, let us mimic the smooth story and carry everything over to the nodal case by replacing K with ω.

Non-Compact Fibers

An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system.

The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters".

Let u_i, θ_i be a system of co-ordinates on the base and fibers such that $\Omega_I = \sum_i du_i \wedge d\theta_i$. If $\theta_{i;C}$ are the co-ordinates for the fiber directions becoming non-compact, then the corresponding $u_{i,C}$ are the co-ordinates on the space of center parameters.

One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Non-Compact Fibers

An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system.

The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters".

Let u_i, θ_i be a system of co-ordinates on the base and fibers such that $\Omega_I = \sum_i du_i \wedge d\theta_i$. If $\theta_{i;C}$ are the co-ordinates for the fiber directions becoming non-compact, then the corresponding $u_{i,C}$ are the co-ordinates on the space of center parameters.

One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Non-Compact Fibers

An important point : The Hitchin map μ is not proper when the base curve C is nodal. Some of the fibers could be non-compact. We want to quotient out these fibers to get a smaller moduli space which is now a Poisson integrable system.

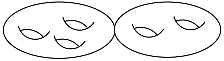
The symplectic duals to these non-compact directions in the fibers remain as Poisson deformation/Casimir parameters. We call them "center parameters".

Let u_i, θ_i be a system of co-ordinates on the base and fibers such that $\Omega_I = \sum_i du_i \wedge d\theta_i$. If $\theta_{i;C}$ are the co-ordinates for the fiber directions becoming non-compact, then the corresponding $u_{i,C}$ are the co-ordinates on the space of center parameters.

One can further freeze (or) set to zero the center parameters to get a symplectic integrable system.

Higgs bundles at a separating node

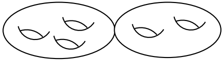
A separating node with $g_C = g_{C_L} + g_{C_R}$:



In this case, the Hitchin I.S factorizes into a left and a right integrable system which share some common compatibility data at the node p. Lets make this explicit.

Higgs bundles at a separating node

A separating node with $g_C = g_{C_L} + g_{C_R}$:



In this case, the Hitchin I.S factorizes into a left and a right integrable system which share some common compatibility data at the node p. Lets make this explicit.

Separating Node

Let us define certain sheaves :

$$\mathcal{L}_k\,|_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L} \quad (3)$$

$$\mathcal{L}_k\,|_{C_R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R} \quad (4)$$

$$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p) \quad (5)$$

$$\mathcal{L}_{k,R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R}(-p) \quad (6)$$

At the node, if we have a non-zero $h^0(\mathcal{L}_k|_{C_L})$ and $h^0(\mathcal{L}_k|_{C_R})$, there is a unique way to glue them together. This data is encoded in the value of a center parameter. One can think of this as the degree k piece of the image under μ of $Res(\phi)_{\{q,r\}}$ where $\{q,r\} = \nu^{-1}(p)$.

Separating Node

Let us define certain sheaves :

$$\mathcal{L}_k \mid_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L} \quad (3)$$
$$\mathcal{L}_k \mid_{C_R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R} \quad (4)$$
$$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p) \quad (5)$$
$$\mathcal{L}_{k,R} = \mathcal{L}_k \otimes \mathcal{O}_{C_R}(-p) \quad (6)$$

At the node, if we have a non-zero $h^0(\mathcal{L}_k|_{C_L})$ and $h^0(\mathcal{L}_k|_{C_R})$, there is a unique way to glue them together. This data is encoded in the value of a center parameter. One can think of this as the degree k piece of the image under μ of $Res(\phi)_{\{q,r\}}$ where $\{q,r\} = \nu^{-1}(p)$.

Separating Node

Let b_k^C be the graded dimension of the space of center parameters. By construction, $b_k^C = 0, 1$ for each k.
Let us also define :

$$B_L := H^0(C, \mathcal{L}_{k,L}) \tag{7}$$

$$B_R := H^0(C, \mathcal{L}_{k,R}) \tag{8}$$

These vector spaces will end up being the bases of the left/right integrable system. Let b_k^L, b_k^R be the corresponding graded dimensions. What are the allowed separating nodes ?

Separating Node

Let b_k^C be the graded dimension of the space of center parameters. By construction, $b_k^C = 0, 1$ for each k.
Let us also define :

$$B_L := H^0(C, \mathcal{L}_{k,L}) \tag{7}$$

$$B_R := H^0(C, \mathcal{L}_{k,R}) \tag{8}$$

These vector spaces will end up being the bases of the left/right integrable system. Let b_k^L, b_k^R be the corresponding graded dimensions. What are the allowed separating nodes ?

Types of Nodes

$\mathcal{L}_k|_{C_L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}$

$\mathcal{L}_{k,L} = \mathcal{L}_k \otimes \mathcal{O}_{C_L}(-p)$

There are 3 possibilities for a T.H.I.S that is OK :

1. **Standard Node** : This is a node in which $h^1(\mathcal{L}_{k,L}), h^1(\mathcal{L}_{k,R}) = 0$ and $b_k^C = 1$ for all k
2. **Regular Restricted Node** : This is a node at which $h^1(\mathcal{L}_{k,L}) > 0$ and/or $h^1(\mathcal{L}_{k,R}) > 0$ for some k and $b_k^C = 0$ for such k.
3. **Non-regular Restricted Node** : This is a node at which $h^1(\mathcal{L}_k|_{C_L}) > 0$ or $h^1(\mathcal{L}_k|_{C_R}) > 0$ for some k and $b_k^C = 0$ for such k. There is a new phenomenon here : In the nodal limit, $h^0(\mathcal{L}_k)$ and $h^1(\mathcal{L}_k)$ increase while keeping $h^0 - h^1$ constant.

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}_{0} + \cancel{b_k^C}_{0} + b_k^R > b_k^{smooth} \tag{9}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.

Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{10}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$b^L_{k\,0} + b^C_{k\,0} + b^R_k > b^{smooth}_k \tag{9}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.

Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{10}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi^{node}_k = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\sout{b_k^L}\,0 + \sout{b_k^C}\,0 + b_k^R > b_k^{smooth} \tag{9}$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.

Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \tag{10}$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

Types of Nodes

For a Non-Regular Restricted Node : If we insist on defining the nodal Hitchin bases as before, we get (say $h^1(\mathcal{L}_k|_{C_L}) = n_k > 0$)

$$\cancel{b_k^L}\,0 + \cancel{b_k^C}\,0 + b_k^R > b_k^{smooth} \qquad (9)$$

Naive definition doesn't give a flat family of integrable systems. But, there is a natural and unique fix.
Define :

$$B_R := H^0(C, \mathcal{L}_{k,R} \otimes \mathcal{O}_{C_R}(-n_k p)) \qquad (10)$$

This modified definition amounts to changing the vanishing orders of $\mathcal{L}_{k,R}$ at p to being $\chi_k^{node} = 1 + n_k$. We do this whenever $h^0(\mathcal{L}_k)$ jumps. Let us put together the values at different k for a generic T.H.I.S ... what are the allowed $(\vec{\chi}, \vec{b^C})$?

Theorem

Theorem In type A tame Hitchin systems that is OK :

- A The vanishing orders χ_k^{node} uniquely determine a nilpotent orbit $\mathcal{O}$.
- B The resulting values of b_k^C are such that the space of center parameters can always be interpreted as the invariant polynomials for some $H \subset SU(N)$, where $H = SU(k), Sp(k)$ for some k.
- C If $[p_i]$ is the partition corresponding to $\mathcal{O}$, then we always have $p_1 - 2 \leq 2\text{rank}(H) \leq 2(p_1 - p_2 - 1)$.

The theorem associates a pair $(\mathcal{O}, H)$ to any node. In this notation, $([N], SU(N))$ is the standard node. Non-separating nodes are always standard.

Theorem

Theorem In type A tame Hitchin systems that is OK :

- A The vanishing orders χ_k^{node} uniquely determine a nilpotent orbit $\mathcal{O}$.
- B The resulting values of b_k^C are such that the space of center parameters can always be interpreted as the invariant polynomials for some $H \subset SU(N)$, where $H = SU(k), Sp(k)$ for some k.
- C If $[p_i]$ is the partition corresponding to $\mathcal{O}$, then we always have $p_1 - 2 \leq 2\text{rank}(H) \leq 2(p_1 - p_2 - 1)$.

The theorem associates a pair $(\mathcal{O}, H)$ to any node. In this notation, $([N], SU(N))$ is the standard node. Non-separating nodes are always standard.

Physical meaning

- The nilpotent orbit $\mathcal{O}$ should be thought of as labelling the defect obtained by taking the OPE of all the defects on C_L
- Part C constrains the defects that can arise on the RHS of such an OPE.
- The H are the different weakly coupled gauge groups that appear in Class $S[A_N]$ theories.

Example

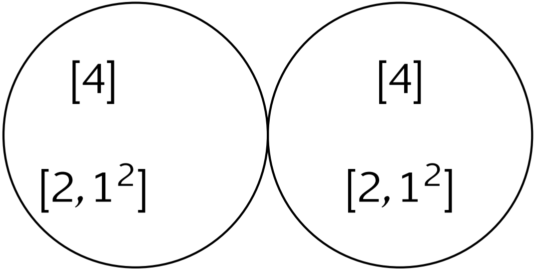
Consider a SL_4 Hitchin system on $C_{0.4}$ with residues in $([4],[4],[2,1^2],[2,1^2])$.
The orders of the zeroes of the ϕ_k are:

E_i	O_H	χ_2	χ_3	χ_4
E_1	$[2,1^2]$	1	2	3
E_2	$[2,1^2]$	1	2	3
E_3	$[4]$	1	1	1
E_4	$[4]$	1	1	1

$\overline{\mathcal{M}_{0,4}}$ has three boundary points.

Example

Couple of them correspond to degenerations of type : (limits correspond to to $SU(4), N_f = 8$ theories)



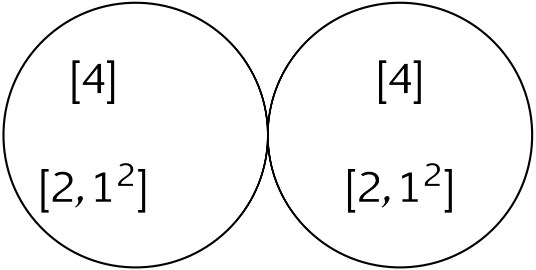
They are both **standard**, ie we would assign the pair $(O, H) = ([N], SU(N))$ to the nodes.

The other node is of the type : (corresponding to a $SU(2)$ gauge theory coupled to a R_4 SCFT + hypers in $\underline{2}$)

This is an example of a **non-regular restricted node**. We assign $(O, H) = ([3, 1], SU(2))$ to this node.

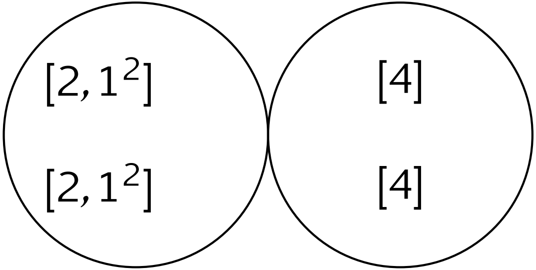
Example

Couple of them correspond to degenerations of type : (limits correspond to to $SU(4), N_f = 8$ theories)



They are both **standard**, ie we would assign the pair $(O, H) = ([N], SU(N))$ to the nodes.

The other node is of the type : (corresponding to a $SU(2)$ gauge theory coupled to a R_4 SCFT + hypers in $\underline{2}$)



This is an example of a **non-regular restricted node**. We assign $(O, H) = ([3, 1], SU(2))$ to this node.

Consequences

- Each part of the theorem was expected from the physics picture but it is reassuring that we could prove them mathematically using our definition of nodal Hitchin system.
- Part A places the notion of OPE of codimension-2 defects on stronger mathematical footing.
- part (C) is a bit stronger as an *a priori* result than what was already known
- Mathematically, this is surprising. Why does χ_k uniquely determine a nilpotent orbit $\mathcal{O}$? And why this list of $(\mathcal{O}, H)$?
- What are the corresponding statements for non-type A Hitchin systems ? At least for the simply laced cases, physics has some non-trivial predictions.

3. Deligne-Simpson

Relation to Deligne-Simpson

The Deligne-Simpson problem :
The problem asks to identify necessary and sufficient conditions on conjugacy classes (in SL_N) $C_1, C_2, \ldots C_{p+1}$ so that we can find matrices $M_i \in C_i$ that obey

$$M_1 M_2 \ldots M_{p+1} = \mathrm{Id} \tag{11}$$

There is also a Lie algebra/additive version.

The Lie Group version is equivalent to asking for non-emptiness of the irreducible character variety on $\mathbb{P}^1$ which is defined to be the moduli space of irreducible representations $\rho : C_{0,p+1} \to SL_N$.

Through the Riemann-Hilbert correspondence and the Non-Abelian Hodge correspondence, this is related to non-emptiness of the moduli space of (semi-)stable Higgs bundles.

Relation to Deligne-Simpson

The Deligne-Simpson problem :
The problem asks to identify necessary and sufficient conditions on conjugacy classes (in SL_N) $C_1, C_2, \ldots C_{p+1}$ so that we can find matrices $M_i \in C_i$ that obey

$$M_1 M_2 \ldots M_{p+1} = \mathrm{Id} \tag{11}$$

There is also a Lie algebra/additive version.

The Lie Group version is equivalent to asking for non-emptiness of the irreducible character variety on $\mathbb{P}^1$ which is defined to be the moduli space of irreducible representations $\rho : C_{0,p+1} \to SL_N$.

Through the Riemann-Hilbert correspondence and the Non-Abelian Hodge correspondence, this is related to non-emptiness of the moduli space of (semi-) stable Higgs bundles.

Relation to Deligne-Simpson

The Deligne-Simpson problem :
The problem asks to identify necessary and sufficient conditions on conjugacy classes (in SL_N) $C_1, C_2, \ldots C_{p+1}$ so that we can find matrices $M_i \in C_i$ that obey

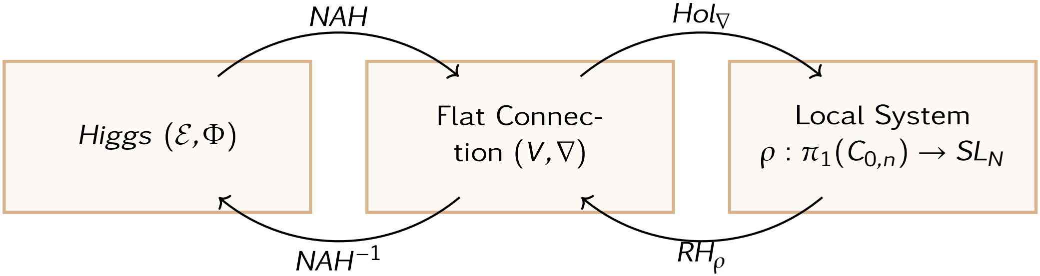
$$M_1 M_2 \ldots M_{p+1} = \mathrm{Id} \tag{11}$$

There is also a Lie algebra/additive version.

The Lie Group version is equivalent to asking for non-emptiness of the irreducible character variety on $\mathbb{P}^1$ which is defined to be the moduli space of irreducible representations $\rho : C_{0,p+1} \to SL_N$.

Through the Riemann-Hilbert correspondence and the Non-Abelian Hodge correspondence, this is related to non-emptiness of the moduli space of (semi-) stable Higgs bundles.

Moduli of Local Systems



NAH involves solving Hitchin's PDE :

$$F_A + [\phi, \phi_h^\dagger] = 0 \quad (12)$$

$$\bar{\partial}_A \phi = 0 \quad (13)$$

Hol is the holonomy map and *RH* is the Riemann-Hilbert map which gives a Fuchsian differential equation with some fixed monodromy data.

Relation to Deligne-Simpson

If the conjugacy classes are semi-simple and when one of the conjugacy classes, say C_{p+1} is regular, Simpson solved the problem in 1992. Let us call this the *regular* case. He gave two conditions for a solution to exist :

1. Let $D = (2g-2)dim(G) + \sum_i dim(C_i)$. Then $D \geq 0$
2. Let $r_a = N - m_a$ where m_a is the largest among the multiplicities of the eigenvalues in any $M_i \in C_i$. Then $\sum_{a=1}^{n} r_a \geq N$

Surprising Result :
Asking for the corresponding Higgs bundle to be OK is the same as asking for Simpson's conditions to hold simultaneously!

Relation to Deligne-Simpson

We arrived at this result via the classification of restricted nodes. This was possible because there is an intimate connection between the existence of a restricted node and bad Hitchin system with atleast one regular conjugacy class :

- Consider a node where C_L is $C_{0,p}$ and C_R is generic
- Now, consider the $p+1$ punctured sphere $C_{0,p+1}$ where we have added a regular/full defect at new puncture.
- Asking for the orginal node in $C_L \cap C_R$ to be restricted $\Longleftrightarrow$ Hitchin system on $C_{0,p+1}$ is bad.

Since we classified the allowed restricted nodes, we also had a classification of all bad (and OK) Hitchin systems with atleast one regular residue.

Relation to Deligne-Simpson

One can just pose the question more directly : Is the OK/Bad dichotomy of Higgs bundles directly related to the existence/non-existence of a solution to the DS problem ?

For the regular cases, we showed that this is indeed the case.

There is strong evidence that this close relationship holds more generally. So, in our paper, we conjectured as much.. with the stronger version of our conjecture saying that the OK condition is necessary and sufficient for the non-emptiness of the character variety.

Relation to Deligne-Simpson

One can just pose the question more directly : Is the OK/Bad dichotomy of Higgs bundles directly related to the existence/non-existence of a solution to the DS problem ?

For the regular cases, we showed that this is indeed the case.

There is strong evidence that this close relationship holds more generally. So, in our paper, we conjectured as much.. with the stronger version of our conjecture saying that the OK condition is necessary and sufficient for the non-emptiness of the character variety.

Relation to Deligne-Simpson

One can just pose the question more directly : Is the OK/Bad dichotomy of Higgs bundles directly related to the existence/non-existence of a solution to the DS problem ?

For the regular cases, we showed that this is indeed the case.

There is strong evidence that this close relationship holds more generally. So, in our paper, we conjectured as much.. with the stronger version of our conjecture saying that the OK condition is necessary and sufficient for the non-emptiness of the character variety.

Relation to Delign-Simpson

The more general cases involve lots of fascinating questions :

- Special role played by rigid local systems esp work of N. Katz (They are Hitchin systems corresponding to free hypers)
- Connections to representations of Quivers and work of W. Crawley-Boevey (via 3d Mirror constructions)

All of these have natural interpretations in the physics setup. (Re-) thinking all of these connections in terms of the OK condition on the line bundles $\mathcal{L}_k$ promises to give a new perspective on the Deligne-Simpson problem.

For ex : In the regular cases, when the Hitchin system is bad, the connection to the nodal curves story furnishes a pair $(\mathcal{O}, H)$ from the corresponding restricted node. Could this pair characterize a reducible local system on $C_{0,p+1}$?

Relation to Delign-Simpson

The more general cases involve lots of fascinating questions :

- Special role played by rigid local systems esp work of N. Katz (They are Hitchin systems corresponding to free hypers)
- Connections to representations of Quivers and work of W. Crawley-Boevey (via 3d Mirror constructions)

All of these have natural interpretations in the physics setup. (Re-) thinking all of these connections in terms of the OK condition on the line bundles $\mathcal{L}_k$ promises to give a new perspective on the Deligne-Simpson problem.

For ex : In the regular cases, when the Hitchin system is bad, the connection to the nodal curves story furnishes a pair $(\mathcal{O}, H)$ from the corresponding restricted node. Could this pair characterize a reducible local system on $C_{0,p+1}$?

Relation to Higgs branches

This appearance of bad Higgs bundles on $C_{L/R}$ implies that the structure group of the underlying principal bundle is reduced.

This property of the **Coulomb branch** should be thought of as analogous to a property that the **Higgs branch** is conjectured to obey (Gaiotto-Moore-Tachikawa). The GMT conjecture says that the smaller groups H should be interpreted as the (hyper-Kähler) isometries of 4d Higgs branches of bad theories $HB(C_{L/R})$.
H. Nakajima has studied the GMT conjecture for the Higgs branch of some of the bad theories and his results are compatible with ours.

Further Applications

- We now have a dictionary between the nodal Hitchin system and $\mathcal{N} = 2$ theories. One can now imagine using a lot of other Class S tools to probe the nodal Hitchin geometry further. For example : Spectral Networks, SUSY indices, partition functions.
- Apply the nodal Hitchin framework to understand constructions of higher Fenchel-Nielsen co-ordinates.
- More generally, can we "bootstrap" the geometry/topology of the Hitchin system using some building blocks + rules for nodal degeneration ?
- The general case of the Deligne-Simpson problem.
- Compactification of the Hitchin fibers in the nodal limit
- Extend all of this to other Cartan types $\mathfrak{j} \in D, E$.
- Extend to wild Hitchin systems