Talk @ ULem on Dec 04, 2018
(based on 1810.10652 or J. Distler)

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3.1.1 Hitchin Systems & 4d $\mathcal{N}=2$ Theories

There is a uniform construction of a large class of 4d $\mathcal{N}=2$ theories from 6d $\mathcal{N}=(1,0)$ theories $X_{\mathcal{E}^j}$ for $j \in 4, 5, 6, 7, 8$.

Theories obtained this way are called “Class S” theories [Gaiotto, GMR].

Idea: $X_{\mathcal{E}^j}$ on $\mathbb{R}^{1,3} \times \mathcal{C}=\mathbb{C}^n$ (Partially twisted)

$4d$ $\mathcal{N}=2$

These 4d $\mathcal{N}=2$ theories have quantum moduli spaces of vacua.

If we have a 4d $\mathcal{N}=2$ SCFT, we typically have different the following two branches:

**Higgs branch**: Where hypermultiplet scalars $\langle q \rangle \neq 0 \& \langle \phi \rangle = 0$

** Coulomb branch**: Where vector multiplet scalars $\langle \phi \rangle \neq 0, \langle q \rangle = 0$. 
What is the Coulomb branch of Class S theories?

\[ \text{Cd} \]
\[ S' \to \sqrt{e_{gm}} \to 4 \text{dN=} 2 \]
\[ S_{dN=} 2 \]
\[ \sqrt{s'} \]
\[ (\ast) \]
\[ e_{gm} \to 3 \text{dN=} 4 \]

The resulting 3d \text{N=} 4 theory has a 4d Coulomb branch.

Preserving \( \frac{1}{2} \) susy in step \((\ast)\) \( \Rightarrow \)

\[ F_A + [\bar{u}, u^+] = 0 \]
\[ \frac{\partial}{\partial \bar{A}} \bar{u} = 0 \] (Hitchin's Eqns)

where \( A = A^{3d}_{C} \) (\( A \) is a 2d \( j \)-gauge field)

\( \bar{u} \) : adjoint valued one-form

Hitchin's Equations have a non-trivial moduli space of solutions: denote that by \( \text{M}_{H} \).

\[ \dim_{C} \text{M}_{H} = \dim (G) (2g-2) \]

\[ \text{S}_{N} : \bar{u} : E \to E \otimes K \text{ (trace free)} \]
Hitchin studied the map:

$$\rho : \mathcal{M}_4 \rightarrow \mathcal{B}$$

$$\mathcal{B} = H^0(C, K \otimes d_i)$$

d_i = 2, 3, 4 \quad \ldots

and showed that \( \mathcal{B} \) is a Lagrangian subspace of \( \mathcal{M}_4 \) and generic fibers of \( \rho \) are complex Lagrangian \( \mathcal{Pl}_0 \).

This is the data underlying a complex integrable system.

Recall: In the 1990s, Seiberg & Witten argued that the low energy EFT on the Coulomb branch can be determined with constraints from SUSY & EM duality of low energy theory (a \( \mathcal{N}=1 \) theory).

They carried this out for \( \text{SU}(2), \text{Nf}=0,1,2,3,4 \).
It was realized later in works of (Donagi–Witten, Freed, Seiberg–Witten) that the geometry underlying the Seiberg–Witten solution of low energy EFT on the constant branch is actually a complex integrable system.

If $f = \sin$, the SW curve is the spectral curve: $\det_{\mathcal{N}} (\psi - \lambda \mathbb{I}) = 0$.

Let $\psi = az^2$ on $\mathbb{T}^4$ so the SW differential is $\psi / z$.

At the punctures, we allow for singular behavior:

$\psi = \frac{a}{z} + (\ldots)$, \texttt{"tame"} \hspace{1cm} $\psi = \frac{a}{z^k} + (\ldots)$, \texttt{"wild"}$, k > 2$.
We'll only study three cases today.

Tame / wild correspond to different classes of defects of the 6d \((0,2)\) theory.

**Remark:** The Hitchin system can be described in 3 different languages:

1. Moduli of Higgs bundles
2. Moduli of flat \(G_{\mathfrak{c}}\)-connections
3. \(\text{Pic} \rightarrow \text{Gr}_{\mathfrak{c}}\) variety

Using solutions to Hitchin's equations, it is possible to go from (I) \(\rightarrow\) (II) (Non Abelian Hodge)

Taking \(H^1(\mathcal{A}_{\text{Pic}})\) on Pic-connection or solving RH problems allows us to go between (II) \(\rightarrow\) (III)

Tame / wild has meaning in all 3 worlds.

I'll stick to Higgs bundles setting.

\(\rightarrow\) can be formulated to other two settings.
If we want a $N=2$ SFT (on the same world), then we want $\text{Res}(\delta) = \alpha$ to be nilpotent.

Let $O_H$ be the adjoint orbit in which $\alpha$ lies. $O_H$ is the "Hidden Orbit" of the defect.

In the presence of such a defect, the local Hitchin moduli space $\mathfrak{g}$ is just the adjoint orbit $O_H$.

Note: We fix the residue to be a fixed orbit.

There are called strongly truncated Hitchin Systems (closely related notions: parabolic Higgs bundles, meromorphic Higgs bundles).

Now, the dimension of the total space increases:

$$\dim (\mathfrak{h}_{\mathfrak{g}H}) = (\dim J)(2g-2) + \sum \dim (\Theta_{iH})$$
§1.2 Two definitions

Once you have a SCFT, it is natural to study relevant deformations of the SCFT. A natural class of deformations are $\mathcal{N}=2$ probe mass deformations. At the $\mathcal{N}=2$ theory has a UV Lagrangian, these correspond to turning on non-zero hyper-multiplet masses. ($\nu_C$ - one complex mass in 4d)

Similar mass deformations exist in 3d $\mathcal{N}=4$ (on $\mathbb{R}^3$, $\mathcal{N}=4$): 3 real masses

We want to study $\nu_C \neq 0$ for the 3d $\mathcal{N}=4$ theory.

What is expected when we turn on non-zero masses?

(a) Coulomb branch dimension does not grow

(b) $\mathcal{N}=2$ into scalar linearly with mass
We encapsulate these expectations in the form of a definition:

**Mass-like deformation (of the same Hitchin system):**

Let \( a_0 \) be \( \text{Res}(\phi) \) before mass deformation and \( a_M \) be \( \text{Res}(\psi) \) after """

Then,
(a) \( \dim(a_{a_0}) = \dim(a_{a_M}) \)
(b) \( [\mathcal{L}] \xrightarrow{m \to 0} \alpha \mathfrak{m} \)

\( [\mathcal{L}] \) is the complex symplectic form on \( \mathcal{M} \) (say, complex structure \( J \)).

Why the "like"?

Turns out that a true mass deformation should obey a further condition:

(associated to every finite defect \( \leftrightarrow \) flavor symmetry group \( F \) (compact Lie group)
\( f: \text{associated complex Lie algebra} \)
Supercosnformal RG theory

\[ \mu \rightarrow \mu + f \] (f - current)

live in the same superconformal multiplet.

(Cordova - Dumitrescu - Intriligator)

This implies \[ \mu_i \in \Lambda(f) \]

This is \( \Rightarrow \) flavor condition.

Mass deformations are those mass-like deformations which obey the flavor condition. (Def 2)

The main result in our paper:

- We classify all mass-like deformations of the affine Hitchin system
- We identify the true mass deformations among the mass-like deformations.

In the rest of the talk, I'll explain this.
§2. Mass-like deformations of the Hitchin system

§2.1 Sheets in a complex Lie algebra

\[ \mathfrak{g}^v \text{ in our case} \]

\[ \mathfrak{U}_d : \text{Union of all adjoint orbits of fixed dimension} \]

Irreducible components of \( \mathfrak{U}_d = \text{sheets} \).

Any sheet has a dense Jordan class in it.

This makes it easy to classify them.

Take a generic element in a sheet

\[ \alpha = g^v \]

\[ \alpha = \mathfrak{ass} + \mathfrak{an} \quad [\mathfrak{ass}, \mathfrak{an}] = 0 \]

Three possibilities:
- purely semi-simple \((\mathfrak{an} = 0)\)
- mixed \((\mathfrak{ass} \neq 0, \mathfrak{an} \neq 0)\)
- purely nilpotent \((\mathfrak{ass} = 0, \mathfrak{an} \neq 0)\)
A sheet is identified by locality $\ell$ via decomposition of generic element $q^n$ and identifying $C_{gr}(ass) = \ell^n$, a locus and $\Theta_{an}$, a nilpotent orbit in $\ell^n$.

$(\ell^n, \Theta) : \text{"sheet label"}$

$C^*$ scaling on eigenvalues $\lambda_{ass}$.

The fixed point of this $C^*$ action is a unique nilpotent orbit (occurs @ $\Theta$ of sheet).

Schematic of a sheet:

\[
\alpha = ass + an
\]
How do we classify sheets in practice? For this, it helps to note the following relation:

\[ \theta_H \text{ occurs at } \omega \text{ of } (L^\nu, \Theta) \]

\[ \iff \]

\[ \mathcal{O}_H = \text{Ind} \mathcal{L}^\nu (\Theta) \]

where \( \text{Ind} \) denotes nilpotent orbit induction in the sense of [Lusztig-Spaltenstein].

We also have

\[ \text{Ind} \mathcal{L}^\nu (\text{Ind} \mathcal{L}^\nu (\Theta)) = \text{Ind} \mathcal{L}^\nu (\Theta) \]

So, to classify sheets, we need to classify pairs \((L^\nu, \Theta_{\text{rigid}})\)

\( \Theta_{\text{rigid}} \): Rigid orbit \( \neq \) never induced from a proper \( \mathcal{L}^\nu < g_{\nu} \).
We have 3 kinds of sheets in a Lie algebra:

**Dixmier Sheets**: \((\mathfrak{g}, 0)\)
- contain semi-simple elements.

**Mixed Sheets**: \((\mathfrak{g}, 0)\)
- generic element is mixed

**Rigid Sheets**: \((\mathfrak{g}, 0)\)
- contains a single rigid nilpotent orbit.

Now, we're ready to go back to the Hitchin system.

**2.2 (A, \phi)** of the Hitchin system in
polar coordinates

\[
A = a(r) d\theta + b(r) \frac{dr}{r}
\]
\[
\phi = b(r) d\theta + c(r) \frac{dr}{r}
\]

check: \(\text{Res} (\phi) = \beta + \text{i} \alpha\)
\(\text{Res} (A) = \alpha\)
$M_{\text{loc}}(\omega, \beta+\gamma) \colon \text{local moduli space}$

eigenvalues of $\omega \colon \mathbb{R}$
eigenvalues of $\beta+\gamma \colon \mathbb{C}$

Now, I pick a sheet in $\mathbb{C}$

$\beta+\gamma$ : any conjugacy class in the sheet.
$\omega$ : real slice of a conjugacy class in sheet.

Let $\mathcal{O}_H$ be the nilp orbit $\subset \mathfrak{g}$ of sheet.

$M_{\text{loc}}(m_R, m_C) \colon \text{parameterization in atoms of matter}$

$[\text{Biquard-Koivalev, Gukov-Witten}]$

$M_{\text{loc}} \xrightarrow{m_R \to 0, m_C \to 0} \overline{\Theta}_H$
That occur to D of Dixmier Sheets $\Rightarrow$ Richardson orbits.

For $\mu R \approx 0$, we a Symplectic Resolution.

$[B, K]: \nu \text{ loc } = \nu \text{ KKS}$

(on fixed adjoint orbit)

On a Dixmier Sheet, varying $\mu C$

leaves underlying real manifold $G_C / Z$

fixed but $\nu C$ scales the $\nu \Sigma$ by $\nu C$

In other words, we've realized that deforming the Higgs residue along a

sheet satisfies all requirements for being a mass-like deformation of the Higgs system.
There is a similar story for non-Richardson orbits

---

\[ G_{\mathbb{C}} / C_{g_{\mathbb{C}}} (A) \]

has residual singularity
\[ Q_{\mathbb{C}} \subset \mathfrak{g} \]
\[ (\subset C_{g_{\mathbb{C}}} (A)) \]

So, to summarise:

Mass-like deformations of the true Hitchin system are classified by sheets in the complex Lie algebra.

[This can be viewed as a systematic generalisation of an observation due to Donagi–Witten for the Hitchin system associated to \( \mathfrak{gs}(n) \) \( n=2m \) utley]
§3. Mass deformations of $T^3(G)$ theories

§3.1 Flavor condition

- We still haven't proposed the flavor condition!

On the one hand, $\mathfrak{g} \in h(F)$

On the other hand, the mass-like parameters $m_c \in Z(\mathfrak{g}^c)$, the center of $\mathfrak{g}^c$.

For a mass-like deformation, to be a mass deformation, we require that

\[
\mathfrak{l}^c = \mathfrak{l}_{BC}^c , \quad \mathfrak{l}_{BC}^c \text{ is Bala-Carter Levi of } \mathfrak{g}.
\]

§3.2 Mass deformed CB of $T^3(G)$ theory
So, how do we identify the mass def\textsuperscript{in} $\mathbf{CB}$ of $T^8\text{[A]}$ theory?

Follow the following algorithm:

1. Find $\mathbf{L}_{BC}$ associated to $\Theta_N$.
   - If $\Theta_N$ is principal Levi type (PL), then look for the sheet $(\mathbf{L}_{BC}, \theta)$ in $\mathfrak{g}_T$. This sheet parameterizes the mass deformed $\Theta_{\mathbf{CB}}$.

2. If $\mathbf{F} \neq 0$ and $\Theta_N$ is not PL type, then calculate $\mathbf{d}_{\mathfrak{g}_T}(\Theta_N) = \Theta_H$
   - $\mathbf{d}_{\mathfrak{g}_T}$: Barbasch-Vogan duality.

[CPT] argued that $\Theta_N \neq \Theta_H$ are related this way.

Now, look at all sheets attached to $\Theta_H$. $\Theta_H$ is always a special orbit in Lusztig's sense.
If any one of the sheets attached to $\Theta_4$ has a label $(c_{BC}, \cdots)$, then such a sheet is unique and this parameterizes the Coulomb branch.

3. If no sheet with $c_{\text{sheet}} = c_{BC}$ occurs, then it is possible to find a unique sub-sheet such that

$$c_{BC} > c_{\text{sheet}}$$

$$\mathcal{C}(c_{BC}) \subset \mathcal{C}(c_{\text{sheet}})$$

4. Finally, if $F=0$, then $\mathcal{L}_B(\Theta_4) = \Theta_4$ if no mass deformation exists.

§3.3 Three deformation classes

1. $\sim$ SMOOTHABLE

2 + 3 $\sim$ MALLEABLE

4 $\sim$ RIGID
A few very interesting observations:

**Definition:** A sheet in a Lie algebra is a special sheet if \( [L,\theta] \) involves a special orbit \( \Theta \) in \( L \).

**Facts:**
1. Special orbits occur at the boundary of at least one special sheet.
2. Special orbits can sometimes occur at the origin of a non-special sheet.

**Interesting Observations:**

A. In steps 2,3, non-special sheets never satisfy the flavor condition (?!)

B. Rigid SCFTs could have non-rigid orbits as their Coulomb branches.