Geometric Langlands from 4d $\mathcal{N} = 2$ theories

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based on 1702.06499 w J. Teschner + work with Ioana Coman-Lohi, J.T

Goals for my talk

Part A : Light introduction to Geometric Langlands (Kapustin-Witten, Beilinson-Drinfeld, Our Motivating Questions)

Part B : Introduce $\mathcal{N} = 2$ Class $\mathcal{S}$ theories (Hitchin system, AGT correspondence)

Part C: Aspects of Geometric Langlands from Class $\mathcal{S}$

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Part A : Approaches to Geometric Langlands

Historical Background

- The Langlands program has its roots in Number Theory, specifically the classification of Automorphic forms for $G(\mathbb{Z})$. These generalize modular forms of $SL(2,\mathbb{Z})$
- For several reasons, it was interesting to ask if there was a geometric analog. That is, a program where only $G(\mathbb{C})$ and its analogs (real forms, loop group etc) would appear and functions on a Riemann surface C replace the global field $\mathbb{Z}$.
- From its early days, some of the program's statements were also known to also have representation theoretic consequences.

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- Familiar classification problems in Rep Theory : Classification of representations of a Finite Group, Classification of unitary representations of a compact Lie group etc.
- In the Langlands story, the relevant representation theory problems involve classification of certain representations (including infinite dimensional) of non-compact Groups.
- The initial program to develop a geometric analog was due to Drinfeld, Drinfeld-Laumon (for specific groups)
- A more general (arbitrary G) and modern program is due to Beilinson-Drinfeld. But, we will take an ahistorical path and instead introduce GL through physics.

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Kapustin-Witten's central idea was that Geometric Langlands can be obtained from properties of a 4d Gauge Theory

- Their starting point was 4d $\mathcal{N} = 4$ SYM with gauge group G and complex coupling τ
- This theory has a (conjectured) S-duality : $(G, \tau) \leftrightarrow (G^\vee, -1/n_s\tau)$, where $G^\vee$ is Langlands/Goddard-Nyuts-Olive dual group. And n_s is the lacing number of the root system associated to $\mathfrak{g}$.
- The electric charge lattice and the magnetic charge lattice are exchanged under S-duality
- S-duality is a non-perturbative duality (exchanges strong and weak coupling)

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More accurately, Kapustin-Witten start with a Topological Quantum Field Theory (TQFT) that is obtained by topological twisting. We deal with the Euclidean version of the theory since this is more natural for TQFT.

- To construct the twisted theory, one exploits the $SO(6)_R$-symmetry
- The twist is defined by providing a particular embedding of $SO(4)' \hookrightarrow SO(4) \times SO(6)$.
- This is called the GL-twisted theory. One actually gets a *family* of TQFTs which, in particular, depends on the complex coupling τ.
- One then studies this TQFT on a particular four manifold $C \times \Sigma$, where C is a genus $g \geq 2$ Riemann surface and Σ is a 2-manifold with a boundary $I \times \mathbb{R}$.

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We now dimensionally reduce the family of TQFTs on C

- In the low energy limit, we get a family of 2d TQFTs which are non-linear sigma models with target being the Hitchin moduli space $\mathcal{M}_H$ (more about the Hitchin system later)
- These are sigma models with $(4, 4)$ SUSY.
- Boundary conditions in $4d$ descend to boundary conditions in the Hitchin sigma model Branes.
- S-duality in the 4d theory acts as T-duality for the 2d Sigma model (Bershadsky et al, Harvey et al)

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The upshot is then :

- S-duality $\sim$ T-duality $\sim$ Mirror Symmetry between A-branes and B-branes in the Hitchin Sigma model $\sim$ Geometric Langlands (SYZ picture of Mirror symmetry plays an important role).
- There is one more important element to the story : We don't just want any A-branes or B-branes, we want branes that obey an Eigenproperty
- This turns out to follow naturally in this TQFT setup (and involves dimensional reduction of 't-Hooft operators in 4d to Hecke operators in the 2d TQFT).
- The B-branes naturally give objects on the Galois side (Electric, $G^{\vee}$) and the A-branes give rise to objects on the Automorphic side (Magnetic, G).

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Beilinson-Drinfeld

Now, how does all this relate to past work on Geometric Langlands ?

- Beilinson-Drinfeld's conjecture has the form of an equivalence of (derived) categories
- Specifically, Beilinson-Drinfeld had the following in mind : $D - mod(Bun_G) \simeq QCohSh(Loc_{G^\vee})$ (do not worry if the terminology is new)
- Here, Bun_G is the moduli space of holomorphic $G_{\mathbb{C}}$ bundles on C. And $Loc_{G^\vee}$ is a pair $(\mathcal{E}, \nabla)$, a holomorphic $G^\vee_{\mathbb{C}}$ - bundle and ∇ is a holomorphic connection.

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- We won't be needing any of the full machinery of derived categories today, but the intuitive idea to remember is a lesson that physicists learnt in the 90s (M. Douglas while studying D-branes in String Theory) : To state "dualities" between extended objects, it is not sufficient to work just with vector spaces.
- This is an equivalence of categories that obeys a Hecke Eigenvalue property. It is similar to what we learn about Matrices, except that the eigenvalue here is a vector space.
- Note that G and $G^\vee$ can be quite different! Ex : $G = SO(2n+1), G^\vee = Sp(2n)$. Even for $G_{\mathbb{C}} = SL(2, \mathbb{C})$, the statement is highly non-trivial.

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Kapustin-Witten and Beilinson-Drinfeld

- How to compare the two approaches ?
- The first thing to note is that both Bun_G and $Loc_{G^\vee}$ are related intimately to the Hitchin system for G and $G^\vee$ respectively.
- On the Electric side, there are known relations between B-branes and Coherent Sheaves. So, this looks promising.
- Furthermore, there is Konstevich's Homological Mirror Symmetry conjecture : It is a duality between a version of Fukaya category (A-model) and Coherent Sheaves (B-model). So, the we are in good shape! (work of Hausel-Thahdeus had also pointed in the direction of a relation between Mirror Symmetry and Langlands duality)

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- One missing puzzle : What about A-branes and D-modules ?
  Kapustin-Witten made an important advance in proposing a way to approach the theory of D-modules using A-branes (work by Nadler-Zaslow, Nadler placed a lot of this on more firm ground)
- So, we seem to have all the pieces of the puzzle!
- But, not quite

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BD approach — - — - — - — - — KW approach

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Closer to CFT

4*d* Gauge Theory

- For Beilinson-Drinfeld, Conformal Field Theory on C played a crucial role. Their D-modules come straight out of CFT play book (differential equations obeyed by WZW conformal blocks)
- What is the link between Kapustin-Witten and CFT ?

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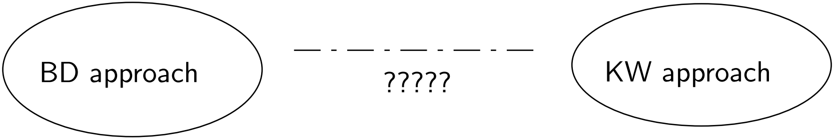

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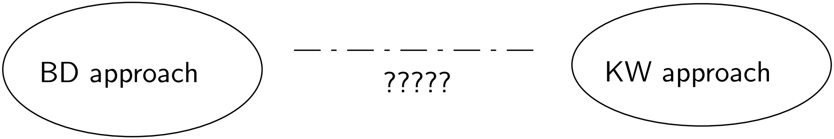

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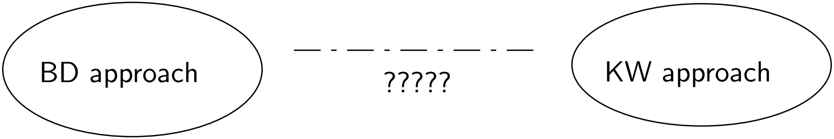

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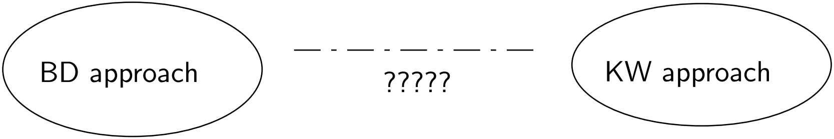

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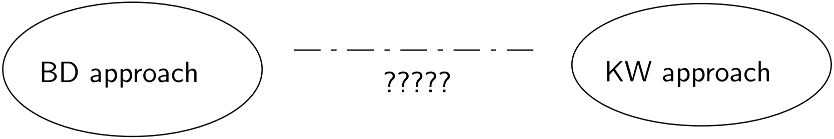

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Part B : Class $\mathcal{S}$ theories

SUSY QFTs in four dimensions

- Number of real supercharges that are possible in four dimensions : 0,4,8,12,16 ($\mathcal{N} = 0, 1, 2, 3, 4$)
- The maximally supersymmetric theory is $\mathcal{N} = 4$. We already encountered it.
- $\mathcal{N} = 3$ theories are of very recent vintage (ex : García Etxebarria, Regaldo)
- $\mathcal{N} = 2$ theories offer an interesting intermediate category : More interesting dynamics compared to $\mathcal{N} = 4$, but still some degree of control over non-perturbative behaviour.
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- Their distinguishing feature is that they admit a construction from the $\mathcal{S}$ix dimensional SCFT with $(0,2)$ SUSY. There is one such theory for every simply laced $\mathfrak{g}$. Sometimes called "Theory $X[\mathfrak{g}]$".
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Class $\mathcal{S}$ theories and the Hitchin System

- Recall that the Coulomb branch geometry of a 4d $\mathcal{N} = 2$ is controlled by a complex integrable system (C.I.S).
- The total space $\mathcal{M}$ of this integrable system is hyper-Kähler (in particular, holomorphic symplectic) manifold that is the Coulomb branch of the 3d theory obtained by reducing the 4d theory on a circle.
- The C.I.S comes equipped with a map $\mu : \mathcal{M} \to \mathcal{B}$, where $\mathcal{B}$ is a half-dimensional base ("Action variables a") and the fibers are complex Lagrangian Tori ("Angle variables θ"). More specifically, there exists co-ordinates in which the h.s form $\Omega = da \wedge d\theta$.

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I can no longer postpone telling you what the Hitchin system is!

- The total space of the Hitchin system $\mathcal{M}_H$ is the moduli space of solutions to a system of PDEs on C :

$$F + [\phi, \phi^\dagger] = 0 \quad (1)$$

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- These are Yang-Mills-Higgs equation for a pair gauge field and an adjoint Higgs : (A, ϕ)
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- This system comes equipped with a map (called Hitchin's first fibration) $\mu_1 : \mathcal{M}_H \to \mathcal{B}$
- Here, $\mathcal{B}$ is the space of Weyl-invariant polynomials built out of $\phi \in \mathfrak{h}(\mathfrak{g})$. For sl_N, locally, $\mathcal{B} = \{Tr(\phi^2), Tr(\phi^3) \ldots\}$. Globally, $\mathcal{B} = \bigoplus_{i=2}^{k} H^0(\Sigma, K^i)$.
- The fibers of μ_1 are complex Lagrangian Tori. Hence the name Hitchin Integrable system.
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- Ex1 : $SU(2), N_f = 4$ is realized as $\mathcal{S}[sl_2, C_{0,4}]$ (with simple poles with residues being the only nilpotent orbit of $\mathfrak{sl}_2$)
- Ex2 : $SU(3), N_f = 6$ is realized as $\mathcal{S}[sl_3, C_{0,4}]$ (simple poles where two of the residues are in the minimal nilpotent orbit and two in the maximal nilpotent orbit).
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AGT correspondence

- One particular observable of Class $\mathcal{S}$ theories, the four sphere partition function $Z_{S^4_{\epsilon_1,\epsilon_2}}$, is particularly relevant for the talk (Localization computation by Pestun, Hama-Hosomichi-Lee).
- This is sensitive to perturbative and non-perturbative physics of the theory.
- A surprising observation of Alday-Gaiotto-Tachikawa : Z_{S^4} is a Liouville correlator on $C_{g,n}$ where $\mathfrak{g} = \mathfrak{sl}_2$ (for specific cases) with $c = 1 + 6(\sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\epsilon_2}{\epsilon_1}})^2$.
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- We will now use the AGT correspondence + its generalizations to obtain aspects of GL from class $\mathcal{S}$ theories.

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Part C : GL and Class $\mathcal{S}$ theories

A view from six dimensions

- One can embed Kapustin-Witten and Class $\mathcal{S}$ into a setup that starts in six dimensions.
- This six dimensional setup gives clues about how to relate KW to 4d $\mathcal{N} = 2$ theories (Witten had some ideas in this direction even before Class $\mathcal{S}$ constructions came into vogue.).
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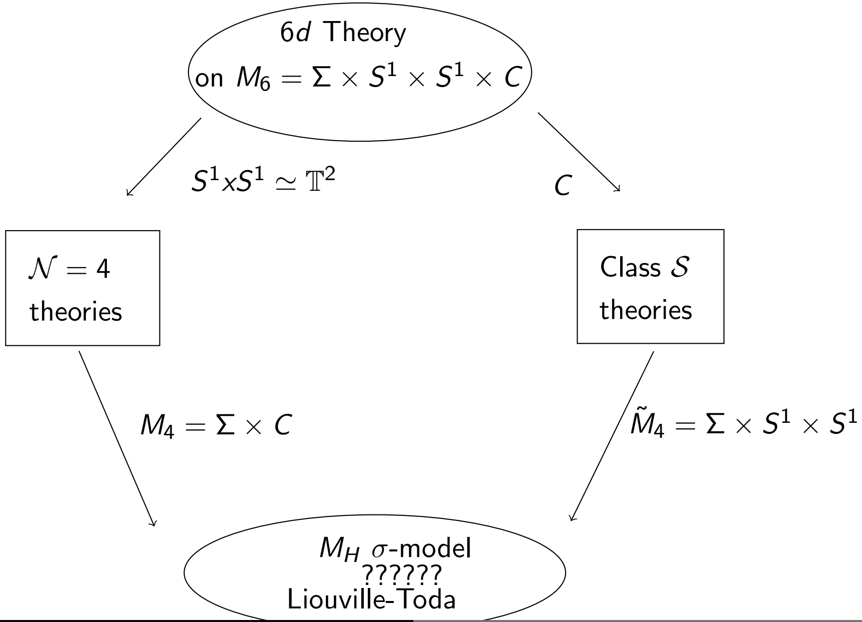
A view of GL from six dimensions



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- AGT only gives Vir confomal blocks. How to obtain (non-rational) WZW conformal blocks from $\mathcal{S}$ theories ? Ans : Calculate Z_{S^4} in the presence of Surface Operators
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What specifically do we do ? We study Class $S[C, g]$ on $S^1_{\epsilon_1} \times S^1_{\epsilon_2} \times \mathbb{T}^2$ under a $\Omega_{\epsilon_1,\epsilon_2}$ deformation and then consider the Nekrasov-Shatashivili (NS) limit $\epsilon_2 \to 0$ (same as critical level limit $k = -2 + \epsilon_1/\epsilon_2 = -2 - b^{-2}$). The, we use the following objects that exist in any Class $\mathcal{S}$ theory :

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More details :

- There is a unique codimension two surface operator that has G global symmetry as a 6d defect.
- This is known from its dimensional reduction on the $\mathcal{N} = 4$ side. In one of the duality frames of $\mathcal{N} = 4$ (with gauge group G), it gives rise to Dirichlet boundary conditions for the gauge field.
- This is one of a family of Nahm-pole boundary conditions for $\mathcal{N} = 4$. This is because of the resulting condition on 3 of the scalars $\overrightarrow{X}$ of $\mathcal{N} = 4$ (bc picks a $SO(3) \in SO(6)_R$).
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- To understand this brane, one has to note that the Hitchin moduli space admits a second description as $T^\star Bun_G$ (crude version, ignoring issues like stability, stacks)
- There is now a natural map (Hitchin's second fibration) $\mu_2 : \mathcal{M}_H \to Bun_G$
- The brane we obtain is a fiber of this map $\mu_2^{-1}(x)$, $x \in Bun_G$. We called it $L_x^{(2)}$ because it is a Lagrangian brane that arises as a fiber of the second Hitchin fibration. This is at one end of the interval I.

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- To relate this sigma model construction to CFT, we use the setup of Nekrasov-Witten (but now with this G-surface operator included)
- They studied Class $\mathcal{S}$ theories on a cigar geometry. We get a cigar geometry in our setup by letting one of the circles shrink to zero size at the end of an interval.
- Nekrasov-Witten gave an argument that $Hom(B_{cc}, B_{opers})$ can be identified with the space of Liouville conformal blocks. (also work of Gaiotto-Witten) . In terms of the second Hitchin fibration, B_{opers} is nothing but μ_2^{-1}(oper bundle).

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- There is a very interesting relationship between WZW conformal blocks and Liouville conformal blocks
- It can be made manifest by a Separation-of-variables (SOV) operation.
- The name is partly motivated by SOV in classical mechanics.
- In the present context, it means finding different set of Darboux co-ordinates (u, v) for the Hitchin system such that $\mathcal{M}_H$ is explicitly given a symmetric product structure : $T^{\star}C^{[n]}$.

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- This is useful as an intermediate step to see an analog of the Hecke Eignevalue property using CFT.
- One can use SOV to represent WZW conformal blocks in terms of Liouville conformal blocks + d $g^\vee$ degenerate fields where $d = 3g - 3$ (Ribault-Teschner, Hikida-Schomerus) In a recent work, Gukov-Frenkel-Teschner gave an interepretation of this in terms of a brane system in M-theory and the creation of $M2$ branes
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GL and Class $\mathcal{S}$: Hecke Property

- In the Nekrasov-Shatashvili limit, the conformal block admits a nice factorization (Teschner). In [A.B,Teschner], we give this a 4d→2d interpretation) ∼ Class $\mathcal{S}$ analog of Hecke Eigenvalue property (but at level of functions!).
- The factorization takes the following form (I have supressed dependence on some parameters) :

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GL and Class $\mathcal{S}$: Which Local System ?

- One aspect that I did not spend a lot of time on : Which local system do we obtain (on the Galois/Electric side) ?
- From the CFT approach, the most natural one to land is the Oper local system. We have an argument of how to obtain this from Kapustin-Witten in our paper (NS limit + Conformal limit on the Galois side).
- But, there is really no reason to restrict to only this type of Local System.
- There is a second set of degenerate fields in the CFT $(\mathfrak{g})$-degenerate fields.
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A clarification on M_H

By Hitchin moduli space, people could refer to three different (but related) models.

- $\mathcal{M}_H^{\zeta=I}$ can be realized as the moduli space of semi-stable Higgs bundles (Dolbeaut)
- $\mathcal{M}_H^{\zeta\neq I}$ can be realized as the moduli space of flat $G_{\mathbb{C}}$-connections (deRham)
- $\mathcal{M}_H^{\zeta\neq I}$ can also be realized as the space of representations of the fundamental group $\pi_1(C) \to G_{\mathbb{C}}$ (Betti)
- One can pass between (Dolbeaut) and (deRham) by solving Hitchin's equations
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- Depending on which model we use for the Hitchin moduli space, we may be in a setting that is complex structure dependent (deRham, Dolbeaut) or complex structure independent (Betti).
- Objects that are algebraic is one setting don't have to be algebraic in another setting. So, one should distinguish between the deRham and Betti algebraic structures.
- So, one may actually want to think of three different Geometric Langlands correspondences : the Dolbeaut (Donagi-Pantev), the deRham (Beilinson-Drinfeld + Arinkin-Gaitsgory), the Betti (Ben-Zvi and Nadler) + ways to go between the three.

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Part D : Bigger Picture/Takeaways

GL and Class $\mathcal{S}$ theories

What are some of the differences to approaching GL using Class $\mathcal{S}$ theories vs approach of Kapustin-Witten ?

The roles of C and $\mathbb{T}^2$ are interchanged. For Kapustin-Witten, $\mathbb{T}^2$ is encoded in non-perturbative physics while C is part of the four dimensional space-time. In the Class $\mathcal{S}$ case, C is encoded in the non-perturbative physics while $\mathbb{T}^2$ in part of 4d space-time.

These differences come with advantages and disadvantages. Also suggests some new questions/connections !

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Further Physics directions

Examples :

- We have a prediction for how certain partition functions in the presence of surface operators should behave.
- In this entire Class S/AGT framework, the natural objects that arise are at the level of functions. These are CFT conformal blocks (or) 4d $\mathcal{N} = 2$ partition functions.
- In the setup relating them to GL, they become geometric analogs of Automorphic functions.
- What does it mean for a 4d Field Theory observable to be a geometric analog of an Automorphic function ?
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Examples :

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- One striking aspect of Class $\mathcal{S}$ theories is that the gauge groups appearing in various duality frames can be quite different (Argyres-Seiberg, Gaiotto, Chacaltana-Distler). The corresponding math problem is the study of the ramified (parabolic) Hitchin system near nodal degenerations of C and GL for nodal curves. Physics may give useful intuition here!

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