Geometric Langlands from 4d $\mathcal{N} = 2$ theories

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based on 1702.06499 w J. Teschner + work with Ioana Coman-Lohi, J.T

Goals for my talk

Part A : Light introduction to Geometric Langlands (Kapustin-Witten, Beilinson-Drinfeld, Our Motivating Questions)

Part B : Review $\mathcal{N} = 2$ Class $\mathcal{S}$ theories (Hitchin system, AGT correspondence)

Part C: Aspects of Geometric Langlands from Class $\mathcal{S}$

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Part A : Approaches to Geometric Langlands

Historical Background

- The Langlands program has its roots in Number Theory, specifically the classification of Automorphic forms for $G(\mathbb{Z})$. These generalize modular forms of $SL(2, \mathbb{Z})$. Fourier co-efficients of Automorphic forms encode very interesting Number Theoretic information.
- For several reasons, it was interesting to ask if there was a geometric analog. That is, a program where only $G(\mathbb{C})$ and its analogs (real forms, loop group etc) would appear and functions on a Riemann surface C replace the global field $\mathbb{Z}$.
- From its early days, some of the program's statements were also known to also have representation theoretic consequences.

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- Familiar classification problems in Rep Theory : Classification of representations of a Finite Group, Classification of unitary representations of a compact Lie group etc.
- In the Langlands story, the relevant representation theory problems involve classification of certain infinite dimensional representations of non-compact Groups (Harmonic Analysis).
- This is a vast generalization of the connections developed between Number Theory and Harmonic Analysis by Gelfand, Harish-Chandra . . ..
- The initial program to develop a geometric analog of the Langlands program was due to Drinfeld, Drinfeld-Laumon (for specific groups)
- A more general (arbitrary G) and modern program is due to Beilinson-Drinfeld. But, we will take an ahistorical path and instead introduce GL through physics.

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Kapustin-Witten's central idea was that Geometric Langlands can be obtained from properties of a 4d Gauge Theory

- Their starting point was 4d $\mathcal{N} = 4$ SYM with gauge group G and complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$
- This theory has a (conjectured) S-duality : $(G, \tau) \leftrightarrow (G^\vee, -1/n_s\tau)$, where $G^\vee$ is Langlands/Goddard-Nyuts-Olive dual group. And n_s is the lacing number of the root system associated to $\mathfrak{g}$.
- The electric charge lattice and the magnetic charge lattice are exchanged under S-duality
- S-duality is a non-perturbative duality (exchanges strong and weak coupling)

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More accurately, Kapustin-Witten start with a Topological Quantum Field Theory (TQFT) that is obtained by topological twisting. We deal with the Euclidean version of the theory since this is more natural for TQFT.

- To construct the twisted theory, one exploits the $SO(6)_R$-symmetry
- The twist is defined by providing a particular embedding of $SO(4)' \hookrightarrow SO(4) \times SO(6)$.
- This is called the GL-twisted theory. One actually gets a *family* of TQFTs which, in particular, depends on the complex coupling τ.
- One then studies this TQFT on a particular four manifold $C \times \Sigma$, where C is a genus $g \geq 2$ Riemann surface and Σ is a 2-manifold with a boundary $I \times \mathbb{R}$.

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We now dimensionally reduce the family of TQFTs on C

- In the low energy limit, we get a family of 2d TQFTs which are non-linear sigma models with target being the Hitchin moduli space $\mathcal{M}_H$ (we will return to the Hitchin system later)
- These are sigma models with $(4,4)$ SUSY.
- Boundary conditions in $4d$ descend to boundary conditions in the Hitchin sigma model Branes.
- S-duality in the 4d theory acts as T-duality for the 2d Sigma model (Bershadsky et al, Harvey et al)

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The upshot is then :

- S-duality $\sim$ T-duality $\sim$ Mirror Symmetry between A-branes and B-branes in the Hitchin Sigma model $\sim$ Geometric Langlands (SYZ picture of Mirror symmetry plays an important role).
- There is one more important element to the story : We don't just want any A-branes or B-branes, we want branes that obey the Hecke Eigenproperty (Aside : Early work on the geometric analog of a Hecke transformation was due to Narasimhan-Ramanan)
- This turns out to follow naturally in this TQFT setup (and involves dimensional reduction of 't-Hooft operators in 4d to Hecke operators in the 2d TQFT).
- The B-branes naturally give objects on the Galois side (Electric, $G^\vee$) and the A-branes give rise to objects on the Automorphic side (Magnetic, G).

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Now, how does all this relate to past work on Geometric Langlands ?

- Beilinson-Drinfeld's conjecture has the form of an equivalence of (derived) categories
- Specifically, Beilinson-Drinfeld had the following in mind : $D - mod(Bun_G) \simeq QCohSh(Loc_{G^\vee})$
- Here, Bun_G is the moduli space of holomorphic $G_{\mathbb{C}}$ bundles on C. And $Loc_{G^\vee}$ is a pair $(\mathcal{E}, \nabla)$, a holomorphic $G^\vee_{\mathbb{C}}$ - bundle and ∇ is a holomorphic connection.

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- We won't be needing any of the full machinery of derived categories today, but the intuitive idea to remember is a lesson that physicists learnt in the 90s (M. Douglas while studying D-branes in String Theory) : To state "dualities" between extended objects, it is not sufficient to work just with vector spaces.
- This is an equivalence of categories that obeys a Hecke Eigenvalue property. It is similar to what we learn about Matrices, except that the eigenvalue here is a vector space.
- Note that G and $G^\vee$ can be quite different! Ex : $G = SO(2n+1), G^\vee = Sp(2n)$. Even for $G_{\mathbb{C}} = SL(2,\mathbb{C})$, the statement is highly non-trivial.

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Kapustin-Witten and Beilinson-Drinfeld

- How to compare the two approaches ?
- The first thing to note is that both Bun_G and $Loc_{G^\vee}$ are related intimately to the Hitchin system for G and $G^\vee$ respectively.
- On the Electric side, there are known relations between B-branes and Coherent Sheaves. So, this looks promising.
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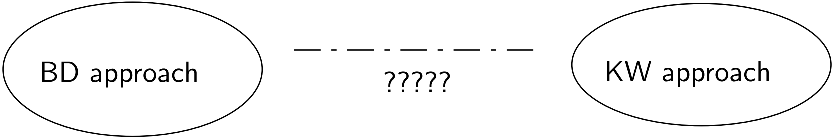

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$4d$ Gauge Theory

- For Beilinson-Drinfeld, Conformal Field Theory on C played a crucial role. Their D-modules come straight out of CFT play book (differential equations obeyed by WZW conformal blocks)
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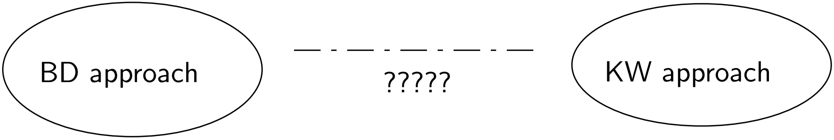

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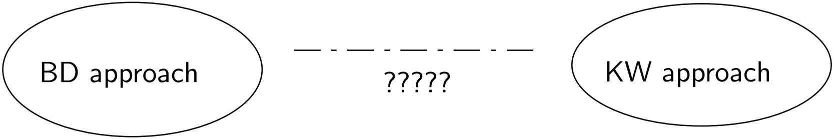

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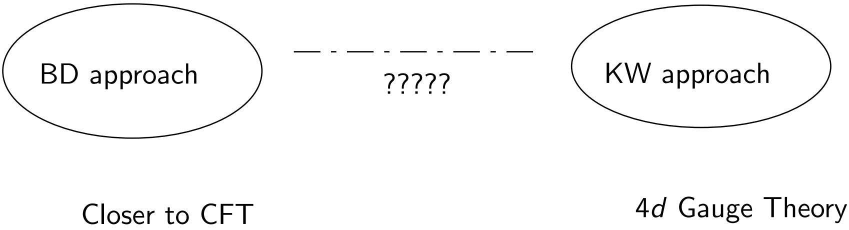

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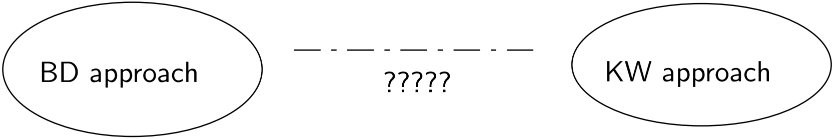

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- Another fundamental issue : Betti (complex structure independent) story vs deRham (complex structure dependent) story (see Ben-Zvi and Nadler).

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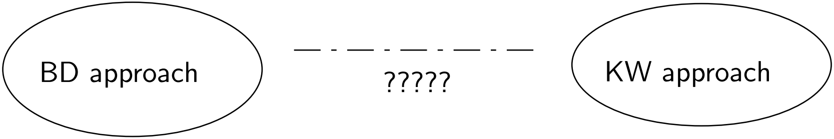

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Part B : Class $\mathcal{S}$ theories

Class $\mathcal{S}$ theories

- One of the (defining) features of $\mathcal{S}$ theories is that the low energy theory on the Coulomb Branch can be described using the Hitchin system for $\mathfrak{g}$ and $C_{g,n}$.
- Familiar Lagrangian theories (including pure $SU(2)$ + $SU(2)$ SQCD with $N_f \leq 4$) arise in this fashion.
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Class $\mathcal{S}$ theories and the Hitchin System

What is the Hitchin System ?

- The total space of the Hitchin system $\mathcal{M}_H$ is the moduli space of solutions to a system of PDEs on C :

$$F + [\phi, \phi^\dagger] = 0 \quad (1)$$
$$\bar{\partial}_A \phi = 0 \quad (2)$$

- These are Yang-Mills-Higgs equation for a pair gauge field and an adjoint Higgs : (A, ϕ)
- $\mathcal{M}_H$ is hyper-Kähler.

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- This system comes equipped with a map (called Hitchin's first fibration) $\mu_1 : \mathcal{M}_H \to \mathcal{B}$
- Here, $\mathcal{B}$ is the space of Weyl-invariant polynomials built out of $\phi \in \mathfrak{h}(\mathfrak{g})$. For sl_N, locally, $\mathcal{B} = \{Tr(\phi^2), Tr(\phi^3) \ldots\}$. Globally, $\mathcal{B} = \bigoplus_{i=2}^{k} H^0(\Sigma, K^i)$.
- The fibers of μ_1 are complex Lagrangian Tori. Hence the name Hitchin Integrable system.
- The defining feature of Class $\mathcal{S}$: Their associated Seiberg-Witten integrable system is the Hitchin integrable system. This encodes several non-perturbative aspects (dualities, BPS spectrum) of the theory in a geometrical fashion.

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- At the punctures of $C_{g,n}$, the Higgs field of the Hitchin system has a pole.
- Ex1 : $SU(2), N_f = 4$ is realized as $\mathcal{S}[sl_2, C_{0,4}]$ (with simple poles with residues being the only nilpotent orbit of $\mathfrak{sl}_2$)
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Part C : GL and Class $\mathcal{S}$ theories

A view from six dimensions

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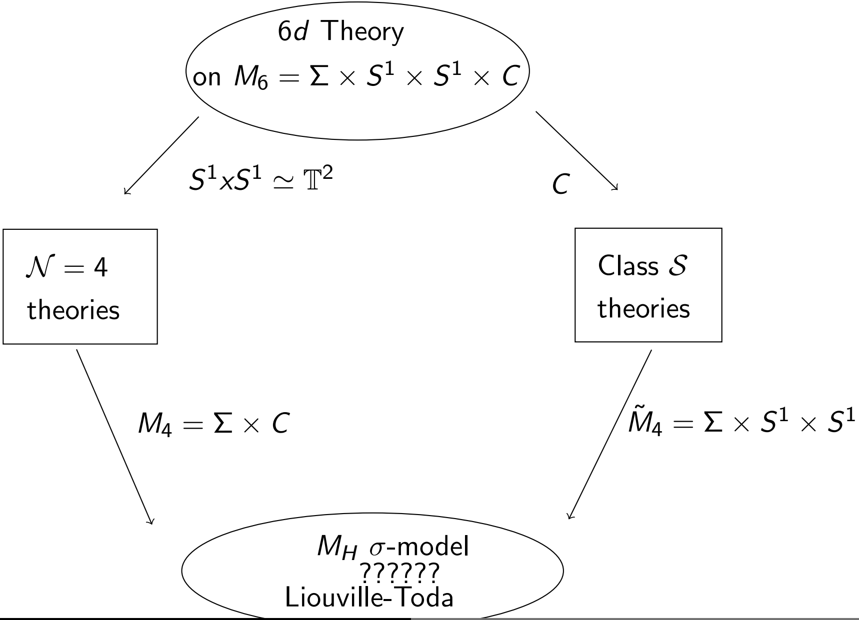

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What specifically do we do ? We study Class $S[C, g]$ on $S^1_{\epsilon_1} \times S^1_{\epsilon_2} \times \mathbb{T}^2$ under a $\Omega_{\epsilon_1,\epsilon_2}$ deformation and then consider the Nekrasov-Shatashivili (NS) limit $\epsilon_2 \to 0$ (same as critical level limit $k = -2 - \epsilon_2/\epsilon_1 = -2 + b^{-2}$). The, we use the following objects that exist in any Class $\mathcal{S}$ theory :

- Surface Operators that arise from codimension-two defects of the 6d theory. We call them co-dimension two surface operators. They wrap C and the $S^1_{\epsilon_2}$ circle. Under AGT, this changes the CFT.
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More details :

- There is a unique codimension two surface operator that has G global symmetry as a 6d defect.
- This is known from its dimensional reduction on the $\mathcal{N} = 4$ side. In one of the duality frames of $\mathcal{N} = 4$ (with gauge group G), it gives rise to Dirichlet boundary conditions for the gauge field.
- This is one of a family of Nahm-pole boundary conditions for $\mathcal{N} = 4$. This is because of the resulting condition on 3 of the scalars $\vec{X}$ of $\mathcal{N} = 4$ (bc picks a $SO(3) \in SO(6)_R$).
- For classical G, Nahm bcs can be brane engineered using $D3 - D5$ branes (Gaiotto-Witten, generalizing work of Diaconescu). In particular, Zero Nahm Pole $\sim$ Dirichlet bcs.

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- To compute the dimensional reduction of the surface operator of Class S, we actually use the $\mathcal{N} = 4$ side and find the dimensional reduction of the zero Nahm pole boundary condition.
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More details :

- To understand this brane, one has to note that an open dense locus of the Hitchin moduli space admits a second description as a cotangent bundle $T^\star Bun^s_G$, where Bun^s_G is the moduli space of stable bundles.
- There is now a natural map (Hitchin's second fibration) $\mu_2 : \mathcal{M}_H \to Bun_G$
- The brane we obtain is a fiber of this map $\mu_2^{-1}(x)$, $x \in Bun_G$. We called it $L^{(2)}_x$ because it is a Lagrangian brane that arises as a fiber of the second Hitchin fibration. This is at one end of the interval I.

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- To relate this sigma model construction to CFT, we use the setup of Nekrasov-Witten (but now with this G-surface operator included)
- They studied Class $\mathcal{S}$ theories on a cigar geometry. We get a cigar geometry in our setup by letting one of the circles shrink to zero size at the end of an interval.
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GL and Class $\mathcal{S}$: A Technical Point

- To precisely fix what the brane L^2_x is, one has to be more careful about stability theory
- The easier case is when $x \in Bun_G^{vs}$, where Bun_G^{vs} is the space of very stable bundles. These are the bundles that do not admit a non-zero Higgs field when viewed as a Higgs bundle.
- When we move away from this locus, the dimension of the space of WZW conformal blocks is expected to jump.
- It is a non-trivial result (due to Laumon) that very stable bundles are stable bundles.
- But, there exist stable bundles that are not very stable. It is interesting to try and study the brane L^2_x and $Hom(B_{cc}, L^2_x)$ over these.

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- There is a very interesting relationship between WZW conformal blocks and Liouville conformal blocks
- It can be made manifest by a Separation-of-variables (SOV) operation.
- The name is partly motivated by SOV in classical mechanics.
- In the present context, it means finding different set of Darboux co-ordinates (u, v) for the Hitchin system such that $\mathcal{M}_H$ is explicitly given a symmetric product structure : $T^{\star}C^{[n]}$.

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- This is useful as an intermediate step to see an analog of the Hecke Eigenvalue property using CFT.
- One can use SOV to represent WZW conformal blocks in terms of Liouville conformal blocks + d $g^\vee$ degenerate fields where $d = 3g - 3$ (Ribault-Teschner, Hikida-Schomerus) In a recent work, Gukov-Frenkel-Teschner gave an interepretation of this in terms of a brane system in M-theory and the creation of $M2$ branes
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- The factorization takes the following form (I have supressed dependence on some parameters) :

$$Z(a, \epsilon_1, \epsilon_2, \tau) \sim_{\epsilon_2 \to 0} e^{-\frac{1}{\epsilon_2}\mathcal{Y}(a,\epsilon_1,\tau)} \Psi(x, a, \tau, \epsilon_1)\psi(a, \tau, \epsilon_1)(1+\mathcal{O}(\epsilon_2)) \quad (3)$$

- The additional piece $\psi(a, \tau, \epsilon_1)$ can be understood as Z_{2d} for a $\mathbb{CP}^1$ sigma model. Coupling to this 2d sigma model is the 4d description of the presence of these additional codimension four surface operators. This corresponds to the simplest possible Hecke Operator for $\mathfrak{sl}_2$.
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- The additional piece $\psi(a,\tau,\epsilon_1)$ can be understood as Z_{2d} for a $\mathbb{CP}^1$ sigma model. Coupling to this 2d sigma model is the 4d description of the presence of these additional codimension four surface operators. This corresponds to the simplest possible Hecke Operator for $\mathfrak{sl}_2$.
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GL and Class $\mathcal{S}$: Which Local System ?

- One aspect that I did not spend a lot of time on : Which local system do we obtain (on the Galois/Electric side) ?
- From the CFT approach, the most natural one to land is the Oper local system. We have an argument of how to obtain this from Kapustin-Witten in our paper (NS limit + Conformal limit on the Galois side).
- But, there is really no reason to restrict to only this type of Local System.
- There is a second set of degenerate fields in the CFT $(\mathfrak{g})$-degenerate fields.
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Betti, deRham and Dolbeaut

By Hitchin moduli space, people could refer to three different (but related) models.

- $\mathcal{M}_H^{\zeta=I}$ can be realized as the moduli space of semi-stable Higgs bundles (Dolbeaut)
- $\mathcal{M}_H^{\zeta\neq I}$ can be realized as the moduli space of flat $G_{\mathbb{C}}$-connections (deRham)
- $\mathcal{M}_H^{\zeta\neq I}$ can also be realized as the space of representations of the fundamental group $\pi_1(C) \to G_{\mathbb{C}}$ (Betti)
- One can pass between (Dolbeaut) and (deRham) by solving Hitchin's equations
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- Depending on which model we use for the Hitchin moduli space, we may be in a setting that is complex structure dependent (deRham, Dolbeaut) or complex structure independent (Betti).
- Objects that are algebraic is one setting don't have to be algebraic in another setting. So, one should distinguish between the deRham and Betti algebraic structures.
- So, one may actually want to think of three different Geometric Langlands correspondences : the Dolbeaut (Donagi-Pantev), the deRham (Beilinson-Drinfeld + Arinkin-Gaitsgory), the Betti (Ben-Zvi and Nadler) + ways to go between the three.

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Part D : Bigger Picture

$\mathcal{N} = 4$ vs Class $\mathcal{S}$ theories

What are some of the differences to approaching GL using Class $\mathcal{S}$ theories vs approach of Kapustin-Witten ?

The roles of C and $\mathbb{T}^2$ are interchanged. For Kapustin-Witten, $\mathbb{T}^2$ is encoded in non-perturbative physics while C is part of the four dimensional space-time. In the Class $\mathcal{S}$ case, C is encoded in the non-perturbative physics while $\mathbb{T}^2$ in part of 4d space-time.

These differences come with advantages and disadvantages. Also suggests some new questions/connections !

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Further Physics directions

Examples :

- We have a prediction for how certain partition functions in the presence of surface operators should behave.
- In this entire Class S/AGT framework, the natural objects that arise are at the level of functions. These are CFT conformal blocks (or) 4d $\mathcal{N} = 2$ partition functions.
- In the setup relating them to GL, they become geometric analogs of Automorphic functions.
- What does it mean for a 4d Field Theory observable to be a geometric analog of an Automorphic function ?
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Examples :

- There is a sense in which class $\mathcal{S}$ theories can be built using building blocks (Gaiotto). CFT counterpart is that one can build N points functions using some 3 point functions + gluing. This gives a Class $\mathcal{S}$ way to think about recent work on Betti version of Geometric Langlands.
- One striking aspect of Class $\mathcal{S}$ theories is that the gauge groups appearing in various duality frames can be quite different (Argyres-Seiberg, Gaiotto, Chacaltana-Distler). The corresponding math problem is the study of the ramified (parabolic) Hitchin system near nodal degenerations of C and GL for nodal curves. Interesting recent work on Higgs bundles on nodal curves by Balaji, Barik and Nagaraj. Physics may give useful intuition here.

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