Geometric Langlands from 4d $\mathcal{N} = 2$ theories

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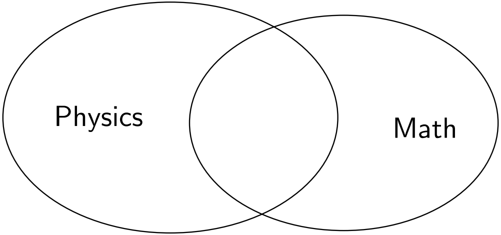
based on 1702.06499 w J. Teschner + work with Ioana Coman-Lohi, J.T

Particles, Strings,
and the Early Universe
Collaborative Research Center SFB 676

Goal for the talk



- Give a very broad introduction to an exciting research area at the interface of Physics and Mathematics.
- An outline of some recent work in this area.

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Somewhat more specifically :

Why are 4d $\mathcal{N} = 2$ theories an interesting laboratory ?

How to obtain aspects of Geometric Langlands/relate approaches in a natural way from certain 4d $\mathcal{N} = 2$ theories

But, I will start with some broad context and motivations.

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- Early 20th century was a period of very active conversation between Quantum Physics and Representation Theory
- What is Representation Theory ? It is the systematic study of Linear actions of Symmetries.
- The background to this interaction was a reorganization of physics with symmetries at the heart Einstein, Lorentz, Noether, . . .
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- With the advent of Quantum Field Theories, this interaction of physics and representation theory only got stronger.
- The crowning glory is, of course, the standard model : It is a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$.
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Why study Supersymmetric Theories ?

- There are several reasons that one may be interested in Supersymmetric QFTs : Beyond SM Physics, Unification, Low energy limits of String/M Theory ..
- The reason that is relevant for this talk : Supersymmetry provides control over non-perturbative aspects of a QFT.
- Ways in which this was realized : Non-renormalization Theorems (perturbative, non-perturbative), Dualities, Supersymmetric Localization.

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$$\beta(g) = -(\frac{11}{3}N_c - \frac{2}{3}N_f)g^2 + (\ldots)g^4 + \ldots \qquad (1)$$

- In contrast, for $\mathcal{N} = 2$ SQCD

$$\beta(g) = -(2N_c - N_f)g^2. \qquad (2)$$

- An analogy : In physics, we often encounter situations in which there exists an idealized problem with extra symmetry + exact solutions. We then hope to arrive at more realistic problems by perturbing this idealized problem. Ex : The two body problem (Integrable) vs N-body problem for $N \geq 3$ (Chaotic).

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SUSY QFTs in four dimensions

- Number of real supercharges that are possible in four dimensions : 0,4,8,12,16 ($\mathcal{N} = 0, 1, 2, 3, 4$)
- The maximally supersymmetric theory is $\mathcal{N} = 4$. The only known theories of this type are SYM theories with some gauge group G. It is also the QFT that has been most amenable to exact computations. Ex : S-duality (More on this later), Large N limits, AdS/CFT, Scattering Amplitudes, Integrable Structures . . ..
- $\mathcal{N} = 3$ theories are of very recent vintage - a lot less is known about them.
- $\mathcal{N} = 2$ theories offer an interesting intermediate category : More interesting dynamics compared to $\mathcal{N} = 4$, but still some degree of control over non-perturbative behaviour.
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Class $\mathcal{S}$ theories

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- Their distinguishing feature is that they admit a construction from $\mathcal{S}$ix dimensional SCFT with $(0, 2)$ SUSY. There is one such theory for every simply laced $\mathfrak{g}$. Sometimes called "Theory $X[\mathfrak{g}]$".
- Dimensional reduction of 6d theory + defects on $C_{g,n} \to$ 4d Class $\mathcal{S}$ theories.
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Class $\mathcal{S}$ theories and CFTs

- Many observables can be calculated exactly in these theories
- One pariticular observable, the four sphere partition function Z_{S^4}, is particularly relevant for the talk.
- This is sensitve to perturbative and non-perturbative physics of the theory.
- A surprising observation of Alday-Gaiotto-Tachikawa : Z_{S^4} is a Liouville correlator on $C_{g,n}$ when $\mathfrak{g} = \mathfrak{sl}_2$. The generalization of this is called the AGT conjecture.

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- The starting points + motivations of each of the approaches are quite distinct. But, it is very interesting that one can land on the same set of problems while having so many different motivations.
- An interesting feature : Each of these approaches involve a particular geometric object : The Hitchin system associated to $C, G_{\mathbb{C}}$.

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- What sort of a statement is Geometric Langlands ?
- It is usually stated as an equivalence of certain (derived) categories . Schematically, it is $D - mod(Bun_G) \simeq Sheaves(Loc - Sys_{G^\vee})$
- At an intuitive level, one should think of it as a (global) analog of Harmonic Analysis that relates some natural objects on the G side to data on the $G^\vee$ side. When $G = U(1)$, this is $\sim$ Fourier Analysis.
- Note that G and $G^\vee$ can be quite different! Ex : $G = SO(2n+1), G^\vee = Sp(2n)$. Even for $G = SU(2)$, the statement is highly non-trivial.

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- Many of the ingredients needed for Geometric Langlands appear naturally in physics (Kapustin-Witten in the context of $\mathcal{N} = 4$). They study a topologically twisted version of $\mathcal{N} = 4$ on $M_4 = \Sigma \times C$ and reduce along C to get a NLSM with target being the Hitchin moduli space associated to G, C.
- When the reduction is enriched to include particular choices of boundary conditions + defects, one obtains suitable "Eigenbranes" in the 2d sigma model. Beilinson-Drinfeld called such objects Hecke Eigensheaves. These are to be thought of as geometric analogs of Automorphic Forms.
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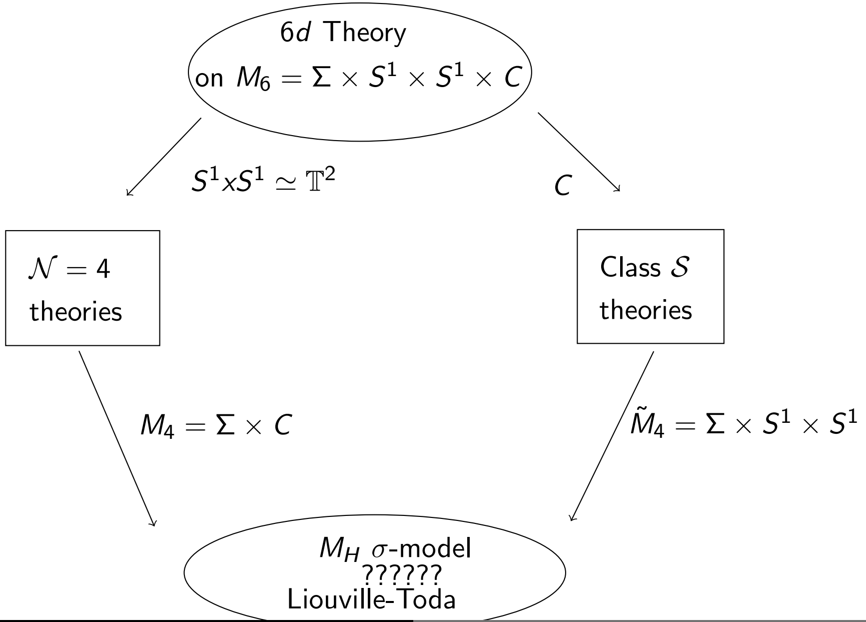

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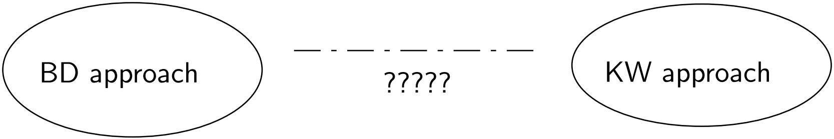

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$4d$ Gauge Theory

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- AGT Conjecture + follow up works gave several heuristic clues that studing Class $\mathcal{S}$ theories under suitable dimensional reductions will help understand relation between [BD] and [KW].
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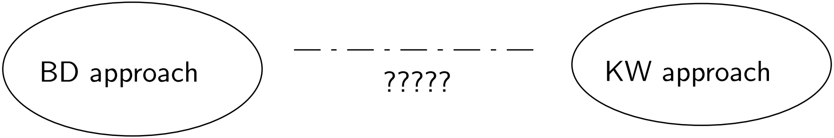

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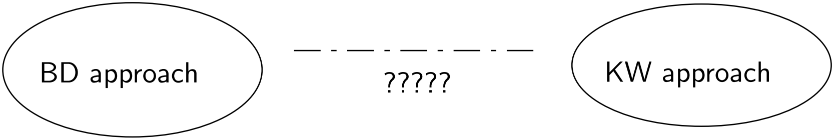

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- Obtain (non-rational) WZW conformal blocks from $\mathcal{S}$ theories ? Ans : Calculate Z_{S^4} in the presence of Surface Operators
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What are some of the differences to approaching GL using Class $\mathcal{S}$ theories vs approach of Kapustin-Witten ?

The roles of C and $\mathbb{T}^2$ are interchanged. For Kapustin-Witten, $\mathbb{T}^2$ is encoded in non-perturbative physics while C is part of the four dimensional space-time. In the Class $\mathcal{S}$ case, C is encoded in the non-perturbative physics while $\mathbb{T}^2$ in part of 4d space-time.

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Examples :

- There is a sense in which class $\mathcal{S}$ theories can be built using building blocks. CFT counterpart is that one can build N points functions using, say, some 3 point functions. This gives a Class $\mathcal{S}$ way to think about recent work on Betti version of Geometric Langlands.
- One striking aspect of Class $\mathcal{S}$ theories is that the gauge groups appearing in various duality frames can be quite different. The corresponding math problem is the study of the ramified (parabolic) Hitchin system near nodal degenerations of C and GL for nodal curves. Physics may give useful intuition here!

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