The Langlands dual group and Electric-Magnetic Duality

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Talk is mostly motivational and historical in nature.

Electric Magnetic Duality in Maxwell's Theory

- Maxwell's equations have an interesting duality : $(E, B) \to (B, -E)$. Equivalently, $(F, \star F) \to (\star F, -F)$.
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- Does this extend to a quantum duality ? Not immediately, since $U(1)$ gauge theories by themselves are not well defined("not UV complete"). But, you could embed the $U(1)$ theory in a non abelian gauge theory. You also obtain non-singular monopoles this way + make sense of a quantum duality.

Duality in $U(1)^n$ gauge theories

- As a preparation for the non-abelian case, consider $G = U(1)^n$ theory.
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- Let the electric charges live in a lattice Γ. Then, Dirac Quantization implies that the magnetic charges necessarily live in the dual lattice $\Gamma^\vee$.

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- More concretely, an electric charge $e_i \in \Gamma = Hom(G, U(1))$ and the magnetic charge $e_i^\vee \in \Gamma^\vee = Hom(U(1), G)$. There is a pairing $\langle e_i, e_j^\vee \rangle = \delta_{ij}$.
- Now, what about non-abelian gauge theories ?

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- Late 70s : ('t-Hooft, Mandelstam, ...) Confining phases of gauge theories like QCD may be described as "EM duals of Higgs phases". Still unclear for theories like QCD but this is known to be a useful viewpoint for understanding phases of certain SUSY Gauge theories that do exhibit phenomenon like confinement, chiral symmetry breaking (Seiberg-Witten).
- General Point : Pay close attention to interplay between Symmetric Breaking and EM Duality.

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- But, in general : NO OBVIOUS ANSWER!! I will give a striking example momentarily.
- To see the subtleties involved, we should learn to distinguish between a Group G and its Langlands/Goddard-Nyuts-Olive dual group $G^\vee$.

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- Let G be a compact Lie group. Let $\mathfrak{g}_{\mathbb{C}}$ be the associated complex Lie algebra. There is a root system associated to $\mathfrak{g}_{\mathbb{C}}$. Denote this by a 4-tuple $R = \{\Lambda_{root}, \Lambda_{weight}, \Lambda_{co-root}, \Lambda_{co-weight}\}$.
- The electric and magnetic charges for the group G are particular objects in this root system.
- Specifically, the electric charges live in the charachter lattice $\Lambda_{char} = Hom(T, U(1))$, where T is a maximal torus of G. In general, $\Lambda_{root} \subset \Lambda_{char} \subset \Lambda_{weight}$.

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- Magnetic charges live in the co-charachter lattice $\Lambda_{co-char} = Hom(U(1), T)$. We also have $\Lambda_{co-root} \subset \Lambda_{co-char} \subset \Lambda_{co-weight}$

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- Most naive expectation for a duality in a non-abelian gauge theory would be for a complete exchange of "electric objects" and "magnetic objects". Say, for example, an exchange of Wilson operators (labelled by Λ_{char} with 't-Hooft operators (labelled by $\Lambda_{co-char}$).

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- In particular, this means an exchange between Λ_{char} and $\Lambda_{co-char}$.
- There is indeed a duality of root systems that also switches Λ_{char} and $\Lambda_{co-char}$. Let this dual root system be $R^\vee$. But, the catch is that it is associated (in general) to a very different Lie algebra $\mathfrak{g}_{\mathbb{C}}^\vee$. There is a corresponding dual group $G^\vee$.
- At the level of groups, some non-trival global properties get exchanged. $\pi_1(G) = Z(G^\vee), Z(G) = \pi_1(G^\vee)$. And we also have $\Lambda_{char}(G) = \Lambda_{co-char}(G^\vee)$ and vice-versa.

Dualities in Non-abelian gauge theories

Examples of Langlands Duality

G	$G^\vee$
$SU(2)$	$SO(3)$
$SU(N)$	$SU(N)/\mathbb{Z}_n$
$SO(2n)$	$SO(2n)$
$Sp(2n)$	$SO(2n+1)$
E_8	E_8

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- It turns out that this most naive expectation of duality can hold only in a very special quantum field theory in four dimensions. This is $\mathcal{N} = 4$ SYM with a gauge group G (with coupling $\tau = \theta/2\pi + 4\pi i/g^2$).

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- There is considerable evidence to support the conjecture ("S-duality") that this theory has exact EM duality and that the dual theory is $\mathcal{N} = 4$ SYM with gauge group $G^\vee$ with coupling $-1/n_r\tau$. (First proposed by Montonen-Olive).

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- One can think of this as the same QFT with two different perturbative limits and both of them admit a description using Lagrangians (albeit different ones).

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- But, note that $H^\vee$ is not a subgroup of $G^\vee$ (!!!!). If there is a relationship between $H^\vee$ gauge theory and the $G^\vee$ gauge theory, this can't be the usual Higgs mechanism for $G^\vee$.
- In the mathematical literature, this feature has a name : "Elliptic Endoscopy".

Geometric Langlands Programs and Quantum Field Theories

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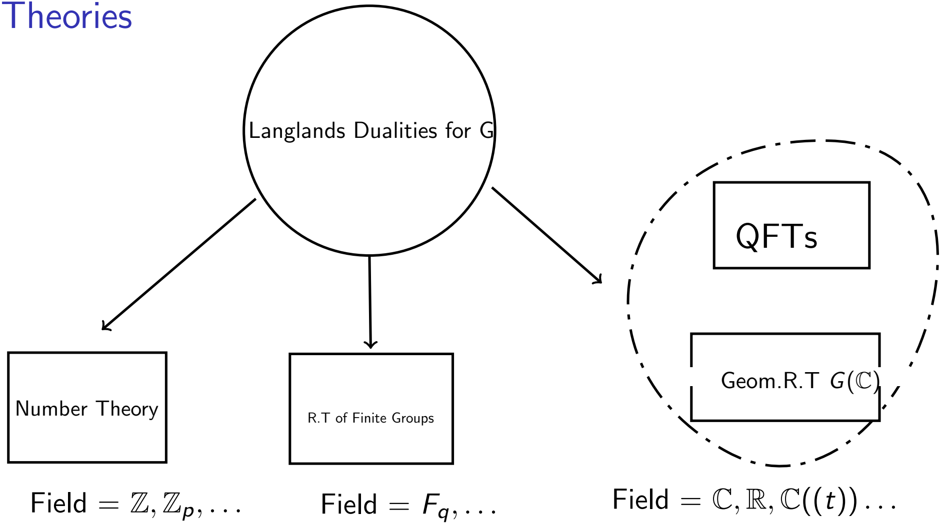
- Langlands' orginal motivation was in a Number Theoretic context : Specifically, to classify Automorphic Forms of a group G (they generalize modular forms of $SL(2, \mathbb{Z})$). His insight was that this classification is intimately tied to properties of the dual group $G^\vee$. He proposed a very detailed program outlining exactly what the relationship is ("Reciprocity", "Functoriality").

Geometric Langlands Programs and Quantum Field Theories



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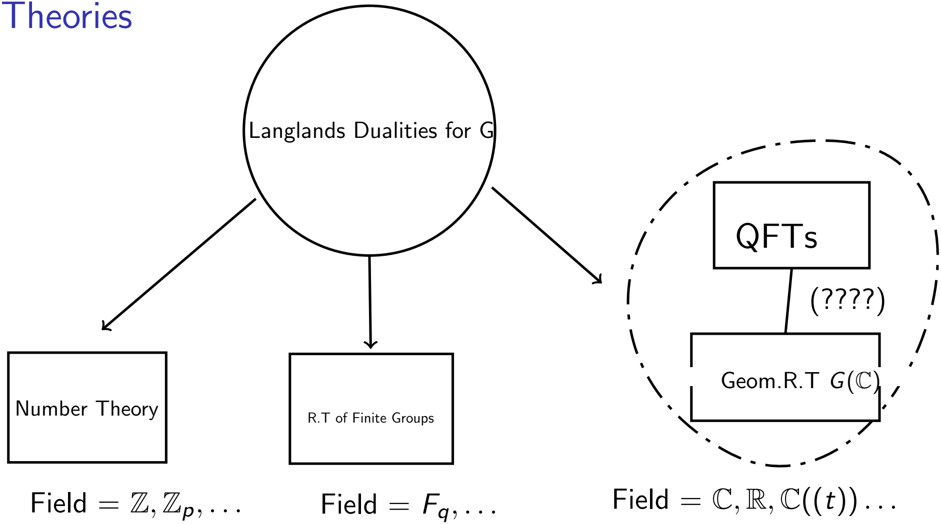

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- In the time since Goddard-Nyuts-Olive, one could have wondered if any of the more detailed features of the Langlands programs appear somewhere in a physics context. One natural case would be to actually think about the Langlands program in the geometric setting ('Geometric' means the Groups are over $\mathbb{C}, \mathbb{C}((t)), \ldots$).

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- A very detailed proposal for a Geometric Langlands program was provided by Beilinson-Drinfeld in the 90s. Curiously, their proposal used some structures that have a striking similarity to (or were almost borrowed from) two dimensional Conformal Field Theory. Is stated as an equivalence of certain Geometric Categories associated to a Riemann Surface C : $Cat(C, G) \cong Cat'(C, G^\vee)$.

Geometric Langlands Programs and Quantum Field Theories

- In 2006, Kapustin-Witten followed a very different path and were able to arrive at the same structures that were important for Beilinson-Drinfeld. Their approach also had several new elements.
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- In particular, Kapustin-Witten's starting point was (a topologically twisted version of) $\mathcal{N} = 4$ Yang Mills theory in four dimensions and its electric magnetic duality.
- When this theory is formulated on a particular four manifold, $M_4 = C \times I_0 \times I_t$, where C is a two dimensional Riemann surface and we study the theory when size of C is very small, then the Geometric Langlands program emerges naturally.
- In particular, the equivalence of categories giving Geometric Langlands is interpreted as a kind of "Mirror Symmetry".

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- The story also has connections to four dimensional theories with lower supersymmetry and theories in other dimensions, including the six dimensional (0,2) SCFT X[j].
- The construction of class $\mathcal{S}$ theories and the associated Alday-Gaiotto-Tachikawa (AGT) type conjectures figure in this bigger picture.
- Lots of unresolved questions : For ex, How to relate the approach of Kapustin-Witten to that of Beilinson-Drinfeld (some initial steps are thanks to Nekrasov-Witten, Teschner, Gaiotto-Witten ..)