Solutions to Quiz 9

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1 A roller and a spring

A roller with mass m and moment of inertia $I = \beta m R^2$ is attached to a wall using a spring (spring constant k). If you pull the roller and let it go, it will roll back and forth. Find the time period T of this motion.

1.1 Solution using energy

The energy of the spring + roller system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 \tag{1}$$

which we can rewrite using $I = \beta M R^2$ and $v = R\omega$ as

$$E = \frac{1}{2}m(1+\beta)v^2 + \frac{1}{2}kx^2 \tag{2}$$

Since the energy is a constant, $dE/dt = 0$, so we take the derivative

$$0 = (1+\beta)mv\frac{dv}{dt} + kx\frac{dx}{dt} \tag{3}$$

But $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = a$. The v cancels, so (3) can be rewritten as

$$a = -\frac{k}{(1+\beta)m}x \tag{4}$$

which is of the form $a = -\omega_0^2 x$. This implies that the motion is simple harmonic, with a time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{(1+\beta)m/k} \tag{5}$$

When $\beta \to 0$ (i.e. we switch off the rolling by setting $I \to 0$), this has the correct limit of the time period of a block and a spring.

1.2 Solution using forces

There are two forces on the roller. Newton's law gives

$$\Sigma F = F_{spring} - F_{friction} = ma \tag{6}$$

The spring force is given by $F_{spring} = -kx$. The torque equation gives

$$\Sigma \tau = F_{friction} R = I\alpha = \beta m R^2 (a/R) \tag{7}$$

which implies that $F_{friction} = \beta ma$. Substituting $F_{friction}$ and F_{spring} into equation (6) gives

$$a = -\frac{k}{(1+\beta)m} x \tag{8}$$

So we get the same result either way. It's always good to have different ways to solve a problem. It turns out that there are many problems which are too complicated to solve using forces, but there are other formulations of mechanics in which these problems become easy.