Here is an outline of the solutions.

1. (i) Expanding \( |\psi\rangle = |1111100\ldots\rangle = c\dagger_5 c\dagger_4 c\dagger_3 c\dagger_2 c\dagger_1 |0\rangle \) we obtain

\[
\begin{align*}
c\dagger_3 c_6 c_4 c_6 \dagger c_3 |\psi\rangle &= c\dagger_3 c_6 c_4 c_6 \dagger c_3 c\dagger_5 c\dagger_4 c\dagger_3 c\dagger_2 c\dagger_1 |0\rangle \\
&= (-1)c\dagger_5 c\dagger_3 c\dagger_2 c\dagger_1 |0\rangle \\
&= -|1110100\ldots\rangle.
\end{align*}
\]

(ii) We may write

\[
|110110010\ldots\rangle = c\dagger_8 |11011000\ldots\rangle = c\dagger_8 c_3 |11111000\ldots\rangle. \tag{1}
\]

This state can be interpreted as the creation of an electron and “hole” in states 8 and 3 respectively.

![Diagram showing electron and hole in states 8 and 3](image)

(iii) To calculate \( \langle \psi | \tilde{N} | \psi \rangle \), where \( \psi = A|100\ldots\rangle + B|111000\ldots\rangle \), note that

\[
\tilde{N} |\psi\rangle = A|100\ldots\rangle + 3B|111000\ldots\rangle, \tag{2}
\]

so that \( \langle \psi | \tilde{N} | \psi \rangle [A^2 + 3B^2] \).

2. (i) We need to confirm that \( \{ c_1, c_\dagger_1 \} = \{ c_1, c_\dagger_1 \} = 1 \) and also \( \{ c_2, c_1 \} = \{ c_2, c_1 \} = 0 \). Substituting for \( c_1 \) and \( c_2 \), we obtain

\[
\{ c_1, c_2 \} = \{ ua_1 + va_\dagger_2, -va_\dagger_1 + ua_2 \} = -uv\{ a_1, a_\dagger_1 \} + vu\{ a_\dagger_2, a_2 \} = 0, \tag{3}
\]
and
\[ \{c_1, c_2^\dagger\} = \{ua_1 + va_1^\dagger, u^*a_1 + v^*a_2\} = |u|^2\{a_1, a_1^\dagger\} + |v|^2\{a_1^\dagger, a_2\} = 1, \] (4)
provided \(|u|^2 + |v|^2 = 1\).

(ii) Consider \(H = \omega[c_1^\dagger c_1 - c_2 c_2^\dagger]\), then if
\[ \left(\begin{array}{c} c_1 \\ c_2^\dagger \end{array}\right) = \left(\begin{array}{cc} u & v \\ -v^* & u^* \end{array}\right) \left(\begin{array}{c} a_1 \\ a_2^\dagger \end{array}\right) = U \left(\begin{array}{c} a_1 \\ a_2^\dagger \end{array}\right), \] (5)
where we note \(U\) is a unitary transformation, we may re-write \(H\) as
\[ H = (c_1^\dagger, c_2)(\omega 0 \quad 0 -\omega)(c_1^\dagger, c_2), \]
so that using (5)
\[ H = (a_1^\dagger, a_2)U^\dagger(\omega 0 \quad 0 -\omega)U(a_1^\dagger, a_2) \]
\[ = \epsilon [a_1^\dagger a_1 - a_2 a_2^\dagger] + [\Delta a_1^\dagger a_2^\dagger + \text{H.c}] \] (6)
where
\[ \epsilon = \omega(|u|^2 - |v|^2), \]
\[ \Delta = \omega u^* v. \] (7)

Squaring both expressions and adding the results, we obtain \(\omega = (\epsilon^2 + \Delta^2)^{\frac{1}{2}}\) and
\[ |u|^2 = \frac{1}{2} \left(1 + \frac{\epsilon}{\omega}\right), \quad |v|^2 = \frac{1}{2} \left(1 - \frac{\epsilon}{\omega}\right), \] (8)

(iii) The ground-state is annihilated by both \(c_1\) and \(c_2\), so that if \(H = \omega[c_1^\dagger c_1 + c_1 c_2^\dagger - 1]\), the ground-state energy is \(E_0 = -\omega = -(\epsilon^2 + \Delta^2)^{\frac{1}{2}}\).

3. Let us write our starting Hamiltonian in the form
\[ H = -\sum_j \left\{ \frac{J_x + J_y}{4} \left( S_{j+1}^+ S_j^- + \text{H.c.} \right) + \frac{J_x - J_y}{4} \left( S_{j+1}^+ S_j^+ + \text{H.c.} \right) \right\}, \] (9)
where \(S^x = S^x \pm S^y\). Using the Jordan Wigner transformation,
\[ S_j^z = (c_j^\dagger c_j - \frac{1}{2}) \]
\[ S_j^+ = c_j^\dagger e^{i\pi \sum_{l<i} \hat{n}_l}, \] (10)
we have
\[ S_{j+1}^+ S_j^- = c_{j+1}^\dagger c_j, \]
so that

\[ H = - \sum_j t[c^\dagger_{j+1}c_j + H.c] - \sum_j \Delta[c^\dagger_{j+1}c_j + H.c] \]  

(12)

where \( t = \frac{J_x + J_y}{4} \), \( \Delta = \frac{J_y - J_x}{4} \).

(ii) Transforming to a momentum basis, \( c^\dagger_j = \frac{1}{N} \sum_q d_q e^{ix_j/q} \), the Hamiltonian takes the form

\[ H = - \sum_{q>0} 2t \cos(qa)[d^\dagger_q d_q + d_{-q} d^\dagger_{-q}] - \sum_q \Delta[e^{-iqa} d^\dagger_q d_{-q} + H.c]. \]  

(13)

Since \( d^\dagger_q d_{-q} = -d_{-q} d^\dagger_q \) is an odd function of \( q \), we can replace \( \Delta e^{-iqa} \rightarrow -2i\Delta \sin qa \), to get

\[ H = \sum_{q>0} \epsilon_q[d^\dagger_q d_q - d_{-q} d^\dagger_{-q}] + \sum_{q>0} i\Delta_q[d^\dagger_q d_{-q} - H.c]. \]  

(14)

where \( \epsilon_q = -2t \cos(qa) \), \( \Delta_q = 2\Delta \sin qa \). Notice how the sum over \( q > 0 \) is needed so that \( d_q \) and \( d_{-q} \) are independent. Carrying out the Bogoliubov transformation

\[
\begin{pmatrix}
  a_q \\
  a^\dagger_{-q}
\end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \begin{pmatrix}
  d_q \\
  d^\dagger_{-q}
\end{pmatrix},
\]

(15)

then following the results of the last section, the Hamiltonian takes the form

\[ H = \sum_{q>0} \omega_q[a^\dagger_q a_q - a_{-q} a^\dagger_{-q}] \]  

(16)

where

\[
\begin{pmatrix}
  u_q \\
  v_q
\end{pmatrix} = \begin{pmatrix}
  1 + \frac{\epsilon_q}{\omega_q} \left[ 1 + \frac{\epsilon_q}{\omega_q} \right]^{-\frac{1}{2}} \\
  i \left[ 1 - \frac{\epsilon_q}{\omega_q} \right]^{-\frac{1}{2}}
\end{pmatrix}
\]

(17)

The spectrum of spin-excitations is shown below. For the case \( J_y = J_x \), the excitation spectrum is gapless, corresponding to the continuous rotational symmetry (Goldstone mode). For the case \( J_y \), or \( J_x = 0 \), the excitation spectrum is flat, as expected for the 1d Ising model.
Figure 2: Showing dispersion for x-y, anisotropic x-y and Ising limits of model.