Intro to Lecture 6

Last time we discussed the idea that a manifold should be described by an atlas of charts, each a 1-1 map of some open set of the manifold into an open set in $\mathbb{R}^n$, with the intrinsic infinitesimal distances on the manifold given on a chart $C_k$ by the covariant metric tensor

$$(ds)^2 = \sum_{j\ell} g_{j\ell} \, dq^j dq^\ell,$$

where $\{dq^j\}$ is a basis of 1-forms in $\mathcal{T}_P^*$, which is called the cotangent space at the point $P$. We found that geodesics, the shortest paths between two points, are described by

$$\ddot{q}^j + \Gamma^j_{mn} \dot{q}^m \dot{q}^n,$$

where $\dot{q}^j = \frac{dq^j}{ds}$, and $\Gamma^j_{mn}$ is the Christoffel symbol given by the metric and its derivatives.

The differential of a scalar field $f : \mathcal{M} \rightarrow \mathbb{R}$ is defined by $df$ and represented in chart $C_k$ by

$$df = \sum_j \frac{\partial \tilde{f}}{\partial q^j} \, dq^j,$$

where $\tilde{f} = f \circ \phi_k^{-1} : C_k \rightarrow \mathbb{R}$. The differential was our first example of a 1-form, but we defined a more general 1-form on $\mathcal{M}$ by

$$\omega(P) = \sum_j \omega_j(\phi_k(P)) \, dq^j$$

where the covariant vector field $\omega_j$ transforms under change of chart $C \rightarrow C'$ as

$$\omega'_j = \frac{\partial q^k}{\partial q'^j} \omega_k.$$

We also found some physical objects are better defined in the dual space, $\mathcal{T}_P$ by contravariant vectors, so the electric field $E_k$ in chart $C$ is transformed so that $Q \cdot \mathbf{E} \cdot dP$, the work done, is intrinsic on the manifold, independent of the choice of chart used to describe $\mathbf{E}$ and $dP$. 
