1. For a series \( \sum_{k=0}^{\infty} a_k \), consider the ratio\( b_k = \left| \frac{a_{k+1}}{a_k} \right| \). If \( \lim_{k \to \infty} b_k = \alpha \) in \( \mathbb{R} \) \((\alpha \neq 0, 1)\), then \( \alpha < 1 \) decides the radius of convergence for our example.

a) \( b_k = \frac{121^k}{(k+1)^2} \). \( \lim_{k \to \infty} b_k = 0 \) for any \( 121 \).

So, this series converges absolutely for any \( z \).

The radius of convergence is \( +\infty \).

b) \( b_k = 121^2 \). \( \lim_{k \to \infty} b_k = 121^2 \).

Converges for \( |121| < 1 \). Radius of convergence is 1.

c) \( b_k = (k+1)^2 121^2 \). \( \lim_{k \to \infty} b_k = +\infty \) for \( z \neq 0 \).

So, only \( z = 0 \) series converges. The radius of convergence is zero.
2. We will take the power series for $e^{2 \cos \theta}$ as the definition of the exponential function.

So

$$
\sum_{n=0}^{\infty} g_n(e^{2 \cos \theta}) = 2 \sum_{n=0}^{\infty} \frac{(2 \cos \theta)^n}{n!}
$$

converges pointwise to $f(\theta) = \frac{1}{2i} e^{2 \cos \theta}$ for each $\theta$, whatever $\theta$ is.

This is also an absolutely convergent series.

$$
|f(\theta) - \sum_{k=0}^{n} g_k(e^{2 \cos \theta})| = \frac{1}{2i} \sum_{k=n+1}^{\infty} \frac{|2 \cos \theta|^k}{k!}
$$

exists and is an upper bound of $|f(\theta) - \sum_{k=0}^{n} g_k(e^{2 \cos \theta})|$

Since $\varepsilon_n(\theta) \to 0$ as $n \to \infty$ \[ \varepsilon_n(\theta) = \frac{1}{2i} \left( 2 \cos \theta - \sum_{k=0}^{n} \frac{|2 \cos \theta|^k}{k!} \right) \] for $\forall \theta > 0$.

We can find an $N(\varepsilon, \delta)$. Let $n > N(\varepsilon, \delta) \Rightarrow |f(\theta) - \sum_{k=0}^{n} g_k(e^{2 \cos \theta})| < \varepsilon$

Note that $N(\varepsilon, \delta)$ does not depend on $\theta$.

Alternative: If a function is given by a power series $P(w) = \sum_{n=0}^{\infty} a_n w^n$, and the radius of convergence of the series is $R$, then
in any region \( |w| < r \), with \( r < R \), the series converges uniformly in \( w \). We could have used it here since the radius of conv. of the exponential function series is \( +\infty \), and \( |2\cos \theta| < |2| < +\infty \).

3. We need show
\[
\frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{2w_0 \theta} d\theta = \sum_{k=0}^{\infty} \frac{2^{2k}}{\pi^k (k!)^2}
\]

[Notice correction!]

Because of uniform convergence, we can use term by term integration, writing
\[
\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f_n(\theta) d\theta
\]
as
\[
\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(\theta) d\theta. \quad \text{Consider an individual term}
\]

\[
\int_{-\pi}^{\pi} \frac{(2\cos \theta)^n}{n!} d\theta = \frac{2^n}{2\pi n!} \int_{-\pi}^{\pi} \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^n d\theta
\]

We can expand \( (e^{i\theta} + e^{-i\theta})^n \) by binomial expansion. The only term that contributes in the integral is \( e^{0i\theta} = 1 \). This happens only if \( n \) is even.

\[
(e^{i\theta} + e^{-i\theta})^{2k} = \sum_{k=0}^{2k} (2k \choose r) (e^{i\theta})^{2k-r} (e^{-i\theta})^r
\]

We have to
pull out the term \( \rho = k \).

So \( \int_{-\pi}^{\pi} g_{2k+1}(\theta) \, d\theta = 0 \)

\[
\int_{-\pi}^{\pi} g_k(\theta) \, d\theta = \frac{e^{2k}}{2\pi(2k)!} \int_{-\pi}^{\pi} \theta^{2k} \frac{e^{\theta}}{\theta} \, d\theta
\]

\[
= \frac{e^{2k}}{(2\pi)^{2k+1} (2k)!} \frac{1}{2k+1} \times (2k)!
\]

\[
= \frac{e^{2k}}{2^{2k} (k!)^2}
\]

So \( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2 \cos \theta} \, d\theta = \sum_{k=0}^{\infty} \frac{e^{2k}}{2^{2k} (k!)^2} \)

4. \( I(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \theta} \, d\theta \)

\[
= \frac{e^x}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \theta} \, d\theta
\]

\( h(\theta) = e^{x (\cos \theta - 1)} \) is an even function and takes the maximum value at \( \theta = 0 \) \( h(0) = 1 \).
b) Plots of $y(\theta) = e^{x(\cos \theta - 1)}$ for $x = 1, 10$ below.

The important observation is that, as $x$ goes to larger values, the peak around $\theta = 0$ gets narrower.
b) \[ I(x) = \frac{e^x}{2\pi} \int_{-1}^{1} e^{x(\cos \theta - 1)} d\theta \]

\[ = \frac{e^x}{\pi} \int_{0}^{\pi} e^{-x(1-\cos \theta)} d\theta \]

Call \( 1-\cos \theta = t, \quad \sin \theta d\theta = dt \)

\[ \sin \theta = \sqrt{1-\cos^2 \theta} = \sqrt{1-(1-t)^2} = \sqrt{2t-t^2} \]

\[ = \sqrt{2t(1-\frac{t^2}{2})} \]

\[ A + \theta = 0 \quad \cos \theta = 1 \quad dt = 0 \]
\[ A + \theta = \pi \quad \cos \theta = -1 \quad dt = 2 \]

\[ I(x) = \frac{e^x}{\pi} \int_{0}^{\pi} e^{-xt} \frac{1}{\sqrt{2t(1-t^2)}} dt \]

\[ = \frac{e^x}{\sqrt{2}\pi} \int_{0}^{\pi} e^{-xt} t^{-\frac{1}{2}} \left(1-\frac{t^2}{2}\right)^{-\frac{1}{2}} dt \]

\[ = \frac{e^x}{\sqrt{2}\pi} \int_{0}^{\infty} e^{-xt} \left(t^{-\frac{1}{2}} + \frac{1}{8} t^\frac{1}{2} + 0(t^{\frac{3}{2}})\right) dt \]

\[ = \frac{e^x}{\sqrt{2}\pi} \left[ \frac{1}{4x} \Gamma \left(\frac{1}{2}\right) + \frac{1}{8} x^\frac{1}{2} \Gamma \left(\frac{3}{2}\right) + O(\frac{1}{x^2}) \right] \]

Using Watson's lemma

\[ = \frac{e^x}{\sqrt{2}\pi \sqrt{x}} \left[ \frac{\Gamma \left(\frac{1}{2}\right)}{4x^\frac{1}{2}} + \frac{1}{8} x^\frac{1}{2} \sqrt{\pi} + O\left(\frac{1}{x^2}\right) \right] \]

\[ = \frac{e^x}{\sqrt{8\sqrt{x}}} \left( 1 + \frac{1}{8x} + O\left(\frac{1}{x^2}\right) \right) \]
So \[ a = \frac{1}{\sqrt{2\pi}} , \quad b = \frac{1}{8\sqrt{2\pi}} \]