1. \[ S = \left\{ t + i \sin \frac{n\pi}{t} \mid t \in (0, 1] \right\} \]

a) Consider the point \( z_0 = 1 + 0i \)

For any \( \varepsilon > 0 \), \( z = 1 + \frac{\varepsilon}{2} + 0i \notin S \) and yet \( |z - z_0| = \frac{\varepsilon}{2} < \varepsilon \)

So, \( S \) is not open.

b) Consider the sequence of points in \( S \) for \( t = \frac{1}{n} \), \( n = 1, 2, 3, \ldots \)

\[ z_n = \frac{1}{n} + i \sin \frac{n\pi}{n} = \frac{1}{n} + 0i \]  Note that \( \{z_n\} \) converges to \( z = 0 + 0i \). Note that \( z \notin S \). If \( S \) was closed any convergent sequence inside \( S \) would have limit in \( S \).

So, \( S \) is not closed.

2. a) It is a series with positive terms. So the partial sums are monotonically increasing.

If the sequence of partial sums are not bounded then the series tends to \( +\infty \).

If the partial sums are bounded, that sequence is convergent, since it is monotonically increasing.
In that case the series is convergent.

b) We apply the integral test. Consider the integral
\[ I = \int_3^\infty \frac{dt}{t (\ln t) (\ln \ln t)} \]

Choose \( s = \ln (\ln t) \) as a substitution.

\[ I = \int_{\ln (\ln 3)}^{\infty} \frac{ds}{s^\alpha} \]

For \( \alpha > 1 \), \( I < +\infty \). So, for \( \alpha > 1 \)

\[ \sum_{n=3}^{\infty} \frac{1}{n (\ln n) (\ln \ln n)^\alpha} \] is convergent

When \( \alpha \leq 1 \), \( I = +\infty \). So, for \( \alpha \leq 1 \),

\[ \sum_{n=3}^{\infty} \frac{1}{n (\ln n) (\ln \ln n)} \] is divergent.

3. We need to prove convergence \( \implies \) Cauchy and Cauchy \( \implies \) convergence.

Let us assume \( \{ x_n \} \to z \). Then for any \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) s.t. \( p > N \] implies
\[ \| x_p - x_N \| < \frac{\varepsilon}{2} \]
Now for \( m, n > N \)
\[
\|x_m - x_n\| = \| (x_m - x) - (x_n - x) \| \\
\leq \|x_m - x\| + \|x_n - x\| \quad \text{Triangle inequality} \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

Hence, we can show that \( \{x_n\} \) is Cauchy.

For the converse, assume \( \{x_n\} \) is Cauchy.
\[
x_n = (\xi_{1n}, \ldots, \xi_{kn}) \in \mathbb{R}^k.
\]

For the \( i \)-th component, \( \{\xi_{in}\} \) is a real sequence.
\[
\text{Since } \quad |\xi_{im} - \xi_{in}| \leq \sqrt{\sum_{j=1}^{k} (\xi_{jm} - \xi_{jn})^2} = \|x_m - x_n\| \\
\{x_n\} \text{ being Cauchy implies } \{\xi_{in}\} \text{ is Cauchy for any } i = 1, \ldots, k. \text{ Hence each of these sequences are convergent. Let us say } \{\xi_{in}\} \rightarrow \xi_i, \text{ and let } x \text{ be } (\xi_1, \ldots, \xi_k).
\]

For any \( \varepsilon > 0 \), there is an \( N_i \in \mathbb{N} \)
\[
\text{s.t. } \quad |\xi_{in} - \xi_i| < \frac{\varepsilon}{\sqrt{k}}, \text{ when } n > N_i.
\]

Choose \( N = \max (N_1, \ldots, N_k) \).
When \( n > N \),
\[
|x_n - x| \leq \frac{1}{k} \sum_{i=1}^{k} |x_i - x_i'|
\]
for all \( i = 1, \ldots, k \).

Thus
\[
\|x_n - x\| = \sqrt{\sum_{i=1}^{k} (x_n - x_i)^2} < \sqrt{\frac{x_1^2}{k} + \cdots + \frac{x_k^2}{k}}
\]
K terms
\[
= \varepsilon
\]

Thus we can show that \( \{x_n\} \) converges to \( x \).

BTW, if you use Bolzano-Weierstrass theorem for \( \mathbb{R}^k \), you get partial credit, since several of proofs of Bolzano-Weierstrass depend on completeness of closed bounded sets, which could depend on Cauchy sequences converging in \( \mathbb{R}^k \).

Bonus Problem: See the graph on the next page.