0.1 Wronskian

A set of $n$ vectors $\vec{V}_j$ is linearly independent if the only set $a_j \in F$ such that $\sum_j a_j \vec{V}_j = 0$ is $a_j = 0$ for all $j$. If the vectors are in an $n$ dimensional space and they are given in terms of a set of basis vectors, we have $\sum_j a_j (V_j)_k = 0$ for all $k$, which means the matrix $M_{kj} = (V_j)_k$ annihilates the vector $a_j$, and the $\vec{V}_j$ are linearly independent if and only if det $M = 0$.

If we have a set of $n$ linearly independent solutions $y_k(x)$ to an $n$th order ordinary linear differential equation, the Wronskian is defined as the determinant of the matrix of $j$th derivatives, $j = 0, \ldots, n-1$ of the $n$ functions $y$,

$$W(x) := \det \frac{d^{\ell-1}y_k}{dx^{\ell-1}} = \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \prod_{j=0}^{n-1} \frac{d^j y_{i_j}}{dx^j},$$

where by $d^k y / dx^k$ we just mean $y$. If we had a set $a_k(x)$ such that $\sum_k a_k(x) \frac{d^k y}{dx^k}(x) = 0$ for all $\ell$, without the $a_k(x)$ all vanishing, we would have a linear dependence among our $n$ solutions, because the $n$th derivative would also vanish, as each $y_k$ satisfies the equation. So linear independence tells us the determinant does not vanish.

If we differentiate $W$, we have

$$\frac{dW}{dx} = \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \prod_{j=0}^{n-1} \frac{d^j y_{i_j}}{dx^j} \prod_{\ell=0}^{n-1} \frac{d^\ell y_{i_\ell}}{dx^\ell}.$$

Note that unless $\ell = n - 1$, the terms in the product with $j = \ell$ and the terms with $j = \ell + 1$ are now identical, except for interchanging the indices on $y$, so the $\epsilon$ kills them, and we have only the contribution from $\ell = n - 1$, which is

$$\frac{dW}{dx} = \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \frac{d^n y_{i_n}}{dx^n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j}.$$

Now if our differential equation is

$$\frac{d^n y}{dx^n}(x) + P(x) \frac{d^{n-1} y}{dx^{n-1}}(x) + \sum_{\ell=0}^{n-2} Q_\ell(x) \frac{d^\ell y}{dx^\ell}(x) = 0$$

we can substitute the values of $d^n y_{i_n}/dx^n$,

$$\frac{dW}{dx} = - \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j} \prod_{\ell=1}^{n-1} \frac{d^\ell y_{i_\ell}}{dx^\ell} (P(x) \frac{d^{n-1} y_{i_n}}{dx^{n-1}}(x) + \sum_{\ell=1}^{n-2} Q_\ell(x) \frac{d^\ell y_{i_n}}{dx^\ell}(x))$$

$$= - \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j} \prod_{\ell=1}^{n-1} \frac{d^\ell y_{i_\ell}}{dx^\ell} \sum_{\ell=1}^{n-2} Q_\ell(x) \frac{d^\ell y_{i_n}}{dx^\ell}(x)$$

$$+ \sum_{i_1, \ldots, i_n} \epsilon_{i_1, \ldots, i_n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j} \prod_{\ell=0}^{n-1} \frac{d^\ell y_{i_\ell}}{dx^\ell} \sum_{\ell=1}^{n-2} Q_\ell(x) \frac{d^\ell y_{i_n}}{dx^\ell}(x)$$

Note that in the terms involving $Q_\ell$, the term before the product matches one of the terms in the product, except for interchange $i_n \leftrightarrow i_\ell$, and so they are killed by the $\epsilon$. Also notice the term multiplying $-P(x)$ is just $W(x)$, so we have the first order differential equation

$$\frac{dW}{dx} = -P(x)W(x).$$