Geometry in Physics

Example 1.
\[
\begin{align*}
\text{Fields in physics:} \\
\text{Vector calculus in Electromagnetism,} \\
\text{Fluid dynamics, vectors and tensors} \\
in elasticity.
\end{align*}
\]

Example 2.
\[
\begin{align*}
\text{Spacetime as 4-dim space} \\
\text{Special and general relativity,} \\
\text{Geodesics and curvature} \\
\text{with metric tensor.}
\end{align*}
\]

Example 3.
\[
\begin{align*}
\text{Classical mechanics} \\
\text{positions and velocities in 6N dimensional space} \\
\text{Position and momenta} \\
(\text{Phase Space})
\end{align*}
\]

We could deal with \( \mathbb{R}^n \) but for some problems we might want to deal with more general "curved" spaces. Hence the motivation to study manifolds.

Example: A space looking like a sphere.
Before we define a manifold we need a few things from point set topology. We have defined open sets and closed sets in \( \mathbb{R}^n \) using a concept of distance. The distance between \( x, y \)

\[
d(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}
\]

allowed definition of \( \varepsilon \)-neighborhoods \( N_\varepsilon(x) = \{y \in \mathbb{R}^n : d(x, y) < \varepsilon\} \).

An open set \( U \) had every point inside it having an \( \varepsilon \)-neighborhood fully inside the open set.

Closed sets were complements of open sets.

These open sets have the property that if \( U_1 \) and \( U_2 \) are open sets \( U_1 \cap U_2 \) is open too.

Proof: For any \( p \in U_1 \), there is an \( \varepsilon_1 > 0 \) s.t. \( N_{\varepsilon_1}(p) \subset U_1 \).

If \( p \in U_2 \), as well, there is an \( \varepsilon_2 > 0 \) s.t. \( N_{\varepsilon_2}(p) \subset U_2 \).

If \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \) then \( N_{\varepsilon}(p) \subset U_1 \cap U_2 \).

Similarly if \( \{U_\alpha \} \) is a family of induced open sets, then \( \bigcup_{\alpha} U_\alpha \) is open as well.

Proof: If \( p \in \bigcup_{\alpha} U_\alpha \), then \( p \in U_\alpha \) for some \( \alpha \in \mathbb{A} \).

\( N_{\varepsilon}(p) \subset U_\alpha \subset \bigcup_{\alpha} U_\alpha \).
The null set $\emptyset$ is open since we cannot find any $p$ violating the condition.

$\mathbb{R}^n$ itself is open, since any point $p \in \mathbb{R}^n$ has an $\varepsilon$-neighborhood inside $\mathbb{R}^n$.

These properties give rise to an axiomatic definition of a topology.

**Def.** A topology on a set $X$ is a set $T$ of subsets of $X$, s.t.

a) If $U_1, U_2 \in T$, then $U_1 \cap U_2 \in T$.

b) If $\{U_k \mid k \in A\} \subseteq T$, then $\bigcup_{k \in A} U_k \in T$.

c) $\emptyset \in T$ and $X \in T$.

The tuple $(X, T)$ is called a topological space. Sometimes $T$ is understood.

**Def.** If $Y \subseteq X$ and $X$ has a topology $T$, we define $T_Y = \{U \cap Y \mid U \in T\}$ the induced (or relative) topology on $Y$.

**Def.** A collection of subsets of $X$, $B$, is a basis of a topology $T$ on $X$ if any open set in $T$ can be formed by unions of sets in $B$. 

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Examples: neighborhoods of points for different $\varepsilon$s for a basis of open sets in $\mathbb{R}^n$.

In fact, if we consider points in $\mathbb{R}^n$ with rational coordinates, and consider open balls with radii which are positive rationals, we get a basis for the topology on $\mathbb{R}^n$ we mentioned. This basis is countable, i.e. one can number these sets as

$$B = \{ U_i : i \in \mathbb{N} \}$$

Def: Spaces with countable bases are called separable.

Def: If $X_1$, $X_2$ has topology $T_1$, $T_2$, then $X_1 \times X_2$ has the product topology generated by the basis $\mathcal{S}_{U_1, U_2}$.

In $\mathbb{R}^n$, in fact, in any metric space a space with a good distance measure, if $x \neq y$, there are non-intersecting open neighborhoods of $x$, $y$.

Def: If $x, y \in X$ with $x \neq y$, the topology $T$ has two open sets $U$ and $V$ s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

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then \((X, T)\) is Hausdorff topological space.

Hausdorff property makes limits unique, for example.

This definition is for completeness. We won't need it so.

**Def.** If a set \(C \subseteq X\) has the \(\text{property that}\)

any open cover of \(C\), namely \([U_s\text{ s.t. } C \subseteq \bigcup_{s} U_s]\),

then there is finite subcover \([U_{s'}\text{ s.t. } C \subseteq \bigcup_{s'} U_{s'}]\).

That is \(C \subseteq \bigcup_{s'} U_{s'}\).

In \(\mathbb{R}^n\) closed and bounded sets are the only compact sets.

**Def.** If \(X, Y\) are topological spaces and

we have \(f : X \rightarrow Y\), and any \(W\) for any open

set \(V\) in \(Y\), \(f^{-1}(W)\) is open in \(X\), then \(f\) is

a continuous function.

**Def.** If \(f : X \rightarrow Y\) is bijection (1-1 and onto) and \(f\) and \(f^{-1}\)

are both continuous, then \(f\) is a homeomorphism.
A homeomorphism says that \( x, y \) are essentially equivalent as topological spaces.

For example:

\[
\text{Subset of } \mathbb{R}^2 \quad \xrightarrow{\phi} \quad \mathbb{R}^2
\]

with induced topology.

Another subset of \( \mathbb{R}^2 \) with induced topology.

Finally, we are ready to define manifolds.

**Def.** If \( M \) is a topological space and \( p \in M \), then a chart at \( p \) is a continuous function \( \phi : U \to \mathbb{R}^n \), with \( p \in U \), and \( \phi \) being a homeomorphism between \( U \) and \( \phi(U) \).

Note \( \phi(x_1, \ldots, x_k) \), the composition in \( \mathbb{R}^n \), are called the coordinates.
As a result, it is often called the coordinate map. The dimension of a chart is $n$. It is some work show it to be unique.

**Def.** A topological manifold is a separable Hausdorff space with an $n$-dimensional chart around any point. The dimension of the manifold is $n$.

In this definition, $\mathbb{R}^n$ is a manifold, but $\mathbb{R}$ is a topological manifold of dimension 1 as well.

We need additional structure to, say, do differential functions on manifolds. We need calculus on manifolds, also called an atlas.

The collection of charts on $\mathbb{M} = \{ (\phi_k, U_k) \}$ sets up maps between open sets in $\mathbb{R}^n$.

Note that $\phi_k \circ \phi_l^{-1} : \phi_l(U \cap U_k) \rightarrow \phi_k(U \cap U_k)$ is a homeomorphism.
$\phi_k \circ \phi_e^{-1}$ is a function from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$.

$\phi_k \circ \phi_e^{-1} : (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n)$

If we demand that these functions have all partial derivatives up to $k$th order (sometimes denoted as $C^k$) we get a related pair of charts.

Def.: An atlas is a $C^k$ atlas if all pairs are $C^k$ related.

Def.: A chart in $C^k$ admissible to an atlas if it is $C^k$ related to all the charts in the atlas.

Def.: A $C^k$ manifold is a topological manifold together with all the admissible charts of some $C^k$ atlas.

For practical purposes, we will be happy with $M$ and an atlas $\mathcal{A} = \{\phi_k \mid k \in \mathbb{Z}^+\}$. We won't need the maximal atlas with all other admissible charts.