Thus \( x = \langle \phi_1, x \rangle \phi_1 + \langle \phi_2, x \rangle \phi_2 + \ldots + \langle \phi_m, x \rangle \phi_m \)

\( \langle \phi_k, x \rangle \phi_k \) is the projection of the vector \( x \) in the direction of \( \phi_k \).

What if \( \{\phi_k\} \) is not necessarily a basis, and consider the projection of a vector \( x \) on the subspace spanned by the system?

**Bessel's Inequality:** \( \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2 \leq \|x\|^2 \)

**Proof:**

\[
\|x\|^2 = \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2 \geq 0
\]

In other words,

\[
\langle x, x \rangle - \sum_{k} \langle \phi_k, x \rangle \overline{\langle \phi_k, x \rangle} \geq 0
\]

\[
\Rightarrow \langle x, x \rangle - \sum_{k} \langle \phi_k, x \rangle \overline{\langle \phi_k, x \rangle} = \sum_{k} \langle \phi_k, x \rangle \overline{\langle \phi_k, x \rangle} \geq 0
\]

\[
\Rightarrow \|x\|^2 - \sum_{k} |\langle \phi_k, x \rangle|^2 \geq 0
\]

\[
\Rightarrow \|x\|^2 \geq \sum_{k} |\langle \phi_k, x \rangle|^2
\]

Q.E.D.
Note that Bessel's inequality is valid even in infinite-dimensional spaces.

One easy consequence of Bessel's inequality is the Cauchy-Schwarz or the Cauchy-Bunyakovsky-Schwartz inequality. Historical note: Cauchy's proof applied to finite-dimensional vector spaces where the other two expanded it to the infinite-dimensional function spaces.

Cauchy-Schwarz Inequality: \( | \langle x, y \rangle | \leq \| x \| \| y \| \)

Proof: IF \( x = 0 \) THEN \( \langle x, y \rangle = 0 \)

\( \langle x+y, y \rangle = \langle y, y \rangle = \| x+y \| + \| y \| = \| y \| \implies \langle x, y \rangle = 0 \)

If \( x \neq 0 \), \( \| x \| > 0 \), Then define \( \phi_x = \frac{1}{\| x \|} x \)

and apply Bessel's inequality to \( y \)

\( | \langle \phi_x, y \rangle |^2 \leq \| y \| \| y \| \)

\( \implies | \langle \frac{x}{\| x \|}, y \rangle |^2 \leq \| y \| \| y \| \) = \( \| \frac{x}{\| x \|} \| \|^2 \leq \| y \| \| y \| \) \implies | \langle x, y \rangle |^2 \leq \| x \| \| y \| \| y \| \implies | \langle x, y \rangle | \leq \| x \| \| y \| \)

\( \implies \langle x, y \rangle \leq \| x \| \| y \| \) by taking sqrt
Cauchy-Schwarz allows us to define an angle $\theta$ between non-zero vectors over reals via
\[
\langle x, y \rangle = \|x\| \|y\| \cos \theta
\]

If you have a set of linearly independent vectors, one can construct an orthonormal system out of it via the Gram-Schmidt orthogonalization process.

\[ \{x_1, \ldots, x_m\} \text{ linearly independent} \] 

(implies they are non-zero vectors)

Idea: Pick the part of $x_i$ orthogonal to the subspace spanned by $\{x_1, \ldots, x_{i-1}\}$.

Precisely:

\[
\phi_i = \frac{x_i}{\|x_i\|}
\]

\[
\tilde{y}_i = x_i - \sum_{k=1}^{i-1} \langle x_k, x_i \rangle \phi_k
\]

Normalize

\[
\phi_k = \frac{\tilde{y}_k}{\|\tilde{y}_k\|}
\]

"Residual"
(We know that \( \| \phi_k \| = 0 \). Otherwise \( \forall k \), \( \phi_k \) would be linearly dependent).

The linear manifold spanned by \( \phi_1, \ldots, \phi_k \) is the same as the one spanned by \( x_1, \ldots, x_k \):

\[ M(\phi_1, \ldots, \phi_k) = M(x_1, \ldots, x_k) \]

**Def.** The orthonormal system \( \phi_1, \phi_2, \ldots \) is complete if \( x \) only vector orthonormal to all the vectors in the system is the zero vector, namely,

\[ \langle \phi_k, x \rangle = 0 \quad \forall k \]

\[ \Rightarrow x = 0 \]

Thus, the orthonormal system \( \phi_1, \ldots, \phi_m \) in an \( n \)-dim vector space \( V \) is complete iff

1. \( m = n \)
2. \( \{\phi_1, \ldots, \phi_m\} \) is a basis

\[ \|x\|^2 = \sum_{k} \langle \phi_k, x \rangle^2 \]

For every pair \( x, y \in V \)

\[ \langle x, y \rangle = \sum_{m} \langle x, \phi_m \rangle \langle \phi_m, y \rangle \]
Proof: Assume \( \phi_1, \phi_2, \ldots, \phi_m \) is a complete orthonormal system.

Consider \( z = x - \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k \).

\[ \langle \phi_k, z \rangle = 0 \quad \text{for} \quad k=1, \ldots, m \]

Hence \( z = 0 \) \( \Rightarrow \) \( x = \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k \).

So any \( x \) can be expanded as a linear combination of \( \phi_k \) since \( \phi_1, \ldots, \phi_m \) are linearly independent. Hence \( \phi_1, \ldots, \phi_m \) in a basis.

So, completeness \( \iff \) (ii)

If \( \phi_1, \ldots, \phi_m \) is a basis and \( V \) is \( n \)-dimensional then \( m = n \).

(ii) \( \Rightarrow \) (i)

If we have an orthonormal system with \( n \) vectors, \( \phi_1, \ldots, \phi_n \) in an \( n \)-dimensional space, we can start with 1 basis \( \phi, \ldots, \phi_3 \) and use the replacement trick to show \( \phi_1, \ldots, \phi_m \) is a basis.

So if \( x = a_1 \phi_1 + \cdots + a_m \phi_m \). Then \( \langle \phi_k, x \rangle = 0 \) for all \( k \).

\[ \therefore a_k = 0 \quad \Rightarrow \quad x = 0 \].

Continued on Page

Read and Understood By

Signed

Date

Signed

Date
So \((i) \implies \text{ completeness}\)

\[
\text{So completeness, (i) } \& \text{ (ii) are equivalent.}
\]

Once more assume completeness \(\mathcal{V}\) the orthonormal system \(\{\Phi_1, \ldots, \Phi_m\}\)

Consider \(x, y \in \mathcal{V}\)

\[
x = \sum_{k=1}^{m} \langle \Phi_k, x \rangle \Phi_k
\]

\[
y = \sum_{k=1}^{m} \langle \Phi_k, y \rangle \Phi_k
\]

\[
\langle x, y \rangle = \left\langle \sum_{k=1}^{m} \langle \Phi_k, x \rangle \Phi_k, \sum_{k=1}^{m} \langle \Phi_k, y \rangle \Phi_k \right\rangle
\]

\[
= \sum_{k=1}^{m} \langle \Phi_k, x \rangle \left\langle \Phi_k, y \right\rangle
\]

\[
= \sum_{k=1}^{m} \langle \Phi_k, x \rangle \langle \Phi_k, y \rangle
\]

\[
= \sum_{k=1}^{m} \langle x, \Phi_k \rangle \langle y, \Phi_k \rangle
\]

So completeness \(\iff (iv)\)

If we \(\Phi_k\) have

\[
\langle x, y \rangle = \sum_{k=1}^{m} \langle x, \Phi_k \rangle \langle \Phi_k, y \rangle
\]

Then, setting, \(y = x\)

\[
\|x\|^2 = \sum_{k=1}^{m} \langle x, \Phi_k \rangle \langle \Phi_k, x \rangle
\]

So \((iv) \implies (iii)\)
Now assume \( \|x\|^2 = \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2 \) for any \( x + y \).

If \( \langle \phi_k, x \rangle = 0 \) for all \( k \),

\[ \|x\|^2 = 0 \quad \Rightarrow \quad x = 0 \]

Hence \( \phi_1, \ldots, \phi_m \) is a complete orthonormal system.

So, (iii) \( \Rightarrow \) Completeness

(iii) \( \Rightarrow \) Completeness \( \Rightarrow \) (ii) \( \Rightarrow \) (i)

Thus, all the statements are equivalent.

Note that if we have a finite dimensional vector space, we have a basis with \( n \) vectors. We can then form an orthonormal basis with \( n \) vectors by the Gram-Schmidt orthogonalization process.
Def: Direct sum of two vector spaces $U$ and $V$ over the same field $F$ could be constructed as follows.

The direct sum $U \oplus V$ consists of ordered pairs $(u, v)$ with $u \in U$, $v \in V$, with addition defined by $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$ and scalar multiple defined by $a \cdot (u, v) = (au, av)$ for $a \in F$. One can show $U \oplus V$ is a vector space.

Def: Sum of vector subspaces $M_1$, $M_2$ of vector space $V$ is defined as

$M_1 + M_2 = \{ x_1 + x_2 | x_1 \in M_1, x_2 \in M_2 \}$.

One can easily show $M_1 + M_2$ is a vector space.

Under some conditions $M_1 + M_2$ is equivalent to $M_1 \oplus M_2$, but these vector spaces are not equivalent in general. The equivalence has to be defined carefully.

(Later!)

One condition of equality is that $M_1 \cap M_2 = \{0\}$.
Def: For vector space $V$ and a subspace $M$, the orthogonal complement $M^\perp$ is defined as $\{x \in V | \langle x, y \rangle = 0 \text{ for all } y \in M \}$.

One can show $M^\perp$ is a subspace.

**Ob 1**

$M \cap M^\perp = \{0\}$

If $x \in M$ and $x + M^\perp$ \[\langle x, x \rangle = 0\]

$\Rightarrow 1 \times 1 = 0 \Rightarrow x = 0.$

(Proof)

Then if $V$ is finite dimensional

$M + M^\perp = V$

If $V$ is finite dimensional then $M$ is finite dimensional. Otherwise, choose a maximal set $I$ of linearly independent vectors in $M$. I would be linearly independent in $V$. Then it cannot have more than $\dim(V)$ elements. So $\dim(M) \leq \dim(V)$.
Now, say \( \dim(M) = m \) and \( \{\phi_1, \cdots, \phi_m\} \)

is an orthonormal basis of \( M \).

For any \( x \in V \), call \( \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k = x' \)

the projection of \( x \) to \( M \). Note that

\[ \langle x - x', \phi_k \rangle = 0 \quad \text{for all} \; k. \]

Since any \( y \in M \) is a linear combination of \( \phi_k \) s

\[ \langle x - x', y \rangle = 0 \quad \text{for all} \; y \in M \]

So \( x - x' = x'' \in M^\perp \)

Hence any \( x \in V \) can be written as

\( x = x' + x'' \) with \( x' \in M \) and \( x'' \in M^\perp \)

So \( M + M^\perp = V \) \( \quad \text{QED} \)

Obs: Since \( MM^\perp = 0 \)

\( M + M^\perp \) is 'equivalent' to \( M \oplus M^\perp \)
We will discuss Product Spaces when we return later.

Sequences of vectors \( \{x_n\} = (x_1, x_2, \ldots) \) with \( x_n \in V \), generalizes our earlier definition on \( \mathbb{R}^k \).

For a normed vector space \( V \), we could define the convergence of sequences, or Cauchy sequences just by realizing \( \mathbb{R}^k \) is one particular norm.

Convergence of \( \{x_n\} \to x \) for vectors means for any \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) s.t. \( n > N \) implies \( \|x_n - x\| < \varepsilon \).

The sequence \( \{x_n\} \) being Cauchy means for any \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) s.t. \( m, n > N \) implies \( \|x_m - x_n\| < \varepsilon \).

For \( \mathbb{R}^k \) or \( \mathbb{C}^n \), these properties are equivalent to some properties for their components. In finite dimensional spaces open up more interesting possibilities.