Some useful tests:

**Comparison test.** Let \( \Sigma x_k \) and \( \Sigma y_k \) be two series of positive numbers. If for some \( N \) we have \( k > N \Rightarrow x_k \geq y_k \) then:

1) if \( \Sigma x_k \) is convergent then \( \Sigma y_k \) is convergent.

2) if \( \Sigma y_k \) is divergent then \( \Sigma x_k \) is divergent.

**Proof.** Simple application of results on monotonic sequences.

**Ratio test.** Let \( \Sigma x_k \) be a series of positive numbers and let \( r_k = \frac{x_{k+1}}{x_k} \).

Then:

1) if only finitely many \( r_k > a \) for some \( 0 < a < 1 \), then the series converges.

2) if only finitely many \( r_k < 1 \) then the series diverges.
An alternative statement

i) If \( \limsup_{k \to \infty} r_k < 1 \) then the series converges.

ii) If \( \liminf_{k \to \infty} r_k > 1 \), then the series diverges.

Proof idea (i): for \( k \) larger than some \( N \), \( r_k < r + \epsilon < 1 \)

\[ \Rightarrow x_k < x_{N+1}(r + \epsilon)^{k-N-1} \text{ for } k \geq N \]

Now use comparison with a geometric series.

Proof idea (ii): Bound \( x_k \) for large \( k \) from below.

Ratio test might run into difficulty if some entries are zero. A related test, namely root test survives there.
Root Test:

Let \( \sum x_k \) be a series on non-negative terms and let \( p_k = k^{\sqrt{x_k}} \).

Then:
1) if only a finite number \( p_k > a \) for some \( a, 0 < a < 1 \), then the series converges.
2) if infinitely many \( p_k > 1 \) then the series diverges.

Closely related statements with \( \limsup \): 

Let \( \limsup p_k = \alpha \)

i) if \( \alpha < 1 \) the series converges
ii) if \( \alpha > 1 \) the series diverges

Note 1: \( \alpha = 1 \) says nothing!
Note 2: The root test statement is only in terms of \( \limsup \).
Note 3: If \( \sum x_k \) is a series with positive terms,

\[
\ln P_n = \frac{1}{n} \ln x_n + \frac{1}{n} \sum_{k=1}^{n} \ln x_k \quad \text{for } n > N
\]

Note 4: \( P_n = x_n \left( \frac{x_{n+1}}{x_n} \right) \ldots \left( \frac{x_{n+1}}{x_1} \right) \)
it turns out that

\[ \liminf_{k \to \infty} \frac{x_{k+1}}{x_k} \leq \liminf_{k \to \infty} x_k^{1/k} \leq \limsup_{k \to \infty} x_k^{1/k} \leq \limsup_{k \to \infty} \frac{x_{k+1}}{x_k} \]

So root test is a finer test.

That result also says that if

\[ \lim_{k \to \infty} \frac{x_{k+1}}{x_k} \]
exists and is in \( \alpha \), then

\[ \lim_{k \to \infty} x_k^{1/k} = \alpha \]

and both tests give the same information.

Simplest example: Geometric series

\( \alpha > 0 \)

\[ \sum_{k=0}^{\infty} \alpha^k = 1 + \alpha + \alpha^2 + \ldots \]

\[ r_k = \frac{x_{k+1}}{x_k} = \alpha \]

\[ r_k = x_k^{1/k} = \alpha \]

Converges when \( \alpha < 1 \), diverges when \( \alpha \geq 1 \)

according to tests. \( \alpha = 1 \) also diverges.
Things get interesting if we consider series of the form $\sum a_k x^k$. More about it later.

However, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$

$$(n^\alpha)^{\frac{1}{n}} = e^{\alpha \cdot \frac{1}{n} \cdot \log n}$$

As $n \to \infty$, $\log n \to 0$

and $(n^\alpha)^{\frac{1}{n}} \to 1$

Trouble spot!

Enter Integral test!

**Integral Test:**

Let $f: [1, \infty) \to \mathbb{R}$ be a continuous, nonnegative, decreasing function and let $x_k = f(k)$ for $k \in \mathbb{N}$.

Then $\sum x_k$ converges iff $I = \int_1^{\infty} f(t) \, dt < \infty$.
Proof idea: Obvious from the picture above.
\[ \sum x_k \text{ dominates } \sum \int_k^{x+1} f(t) dt \quad \text{for all } k \]
\[ \sum' x_k \text{ is dominated by } \sum \int_k^{x+1} f(t-1) dt \quad \text{for all } k \geq 1 \]

Application to \( \sum_{n=1}^{\infty} \frac{1}{n^r} \)

\[ I(m) \int_0^m \frac{dt}{e^t} = \frac{1 - e^{-m}}{1 - e^{-1}} \quad \text{for } 0 \neq 1 \]

\[ = \log m \quad \text{for } 0 = 1 \]

As \( M \to \infty \), \( I(m) \) is bounded iff \( 0 > 1 \).

So \( \sum \frac{1}{n^r} \) converges iff \( 0 > 1 \).
Alternating series and rearrangements.

We have dealt with many series where we have absolute convergence. Let us now look at some conditionally convergent series.

Let \( \{x_k\} \) be a sequence of nonnegative numbers. Define the alternating series as

\[ S = \sum_{k=0}^{\infty} (-1)^k x_k \]

Thm: For decreasing \( \{x_k\}\)

\[ S \text{ converges iff } \{x_k\} \to 0 \]

Proof idea: \( x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0 \)

Consider partial sums \( S_n = \sum_{k=0}^{n} (-1)^k x_k \)

Essentially odd \( S_n\):

\( S_n \) is a decreasing sequence. For \( n \) even \( S_n \)'s make an
increasing sequence \( s_{2m+1} - s_{2m} = x_{2m} \)

As \( m \to \infty \), this difference goes to zero.

Example: Take \( x_k = \frac{1}{k+1} \quad k = 0, 1, 2, \ldots \)

We know \( \sum_{k=0}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \ldots \) the harmonic series diverges.

However, \( \sum_{k=0}^{\infty} (-1)^k x_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) converges conditionally. In fact, it converges to \( \ln 2 \).

Rearrangements: Let \( s_1, s_2, s_3, \ldots \) be any onto mapping to \( \mathbb{N} \).

The sequence \( s_k s_1, s_2, s_3, \ldots \) is a rearrangement of \( s_k \).

What happens when one rearranges a series?

For a convergent series, it matters if it is absolutely convergent or not.

For absolutely convergent series, rearrangements lead to the same limit. Not so for conditionally convergent.
Example: Alternating harmonic series

\[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots\] goes to ln2.

Now consider the rearrangement

\[1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots\]

What is the limit here?

If we club doublets in the first series

\[(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6})\]

\[= \frac{1}{2m+1} - \frac{1}{2m+2}\]

The series \(\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}\) converges by integral test. It turns out to have the same limit as the alternating harmonic series.

For the rearrangement, collect triplets together

\[(1 + \frac{1}{3} - \frac{1}{2}) + (\frac{1}{5} + \frac{1}{7} - \frac{1}{4}) + \ldots\]
\[
\left( \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \right) \quad \text{for } n = 0, 1, 2, \ldots
\]

\[
\sum_{n=0}^\infty \frac{8n+5}{2(n+1)(4n+1)(4n+3)}
\]

This also converge by integral test.

However, \( \frac{1}{4n+1} + \frac{1}{4n+3} \) is bigger than \( \frac{1}{2n+2} \).

[Arithmetic mean bigger than harmonic mean.]

More directly,

\[
\frac{1}{4n+1} + \frac{1}{4n+3} = \frac{8n+4}{(4n+1)(4n+3)} = \frac{4(2n+1)}{(4n+1)(4n+3)}
\]

\[
= \frac{4(2n+1)}{4n+2-1} \cdot \frac{1}{(4n+3)} = \frac{4(2n+1)}{4(2n+1)^2 - 1} \cdot \frac{1}{4(2n+1)^2 - 1}
\]

\[
> \frac{4(2n+1)}{4(2n+1)^2} = \frac{1}{2n+1}
\]

So this series converges to a higher value than \( \ln 2 \).

In fact, Riemann series theorem says...
that for a conditionally convergent series, there are rearrangements that make it converge to any desired number. There are also rearrangements that can make it go to $+\infty$ or $-\infty$.

Let us finish with some comments on infinite products of the form

$$\prod_{m=1}^{\infty} \left(1 + z_m \right), \text{ with partial products } p_n = \prod_{m=1}^{n} \left(1 + z_m \right).$$

Convergence is decided by convergence of the sequence $\{p_n\}$. Absolute convergence has to satisfy

$$\prod_{m=1}^{\infty} (1 + |z_m|).$$

For non-negative $\{z_m\}$,

$$\prod_{m=1}^{\infty} (1 + z_m) \text{ converges iff } \sum_{m=1}^{\infty} z_m \text{ converges.}

\text{Proof: This follows for } \sum_{m=1}^{n} z_m < \prod_{m=1}^{n} \left(1 + z_m \right) < \prod_{m=1}^{n} e^{z_m} = e^{\sum_{m=1}^{n} z_m}$$

and that partial products are increasing.
By the absolute convergence guarantees convergence

\[ p_n = \prod_{k=1}^{n} (1 + z_k) \quad q_n = \prod_{k=1}^{n} (1 + z_k) \]

Note that \[ |p_n| \leq |q_n| \]
\[ |p_m - p_n| = |p_n| \prod_{k=n+1}^{m} (1 + z_k) - 1 | \quad \text{for } m > n \]

By triangle inequality,
\[ \prod_{k=n+1}^{m} (1 + z_k) - 1 + 1 \geq \prod_{k=n+1}^{m} (1 + z_k) - 1 + 1 \]

or
\[ \prod_{k=n+1}^{m} (1 + z_k) - 1 \leq \prod_{k=n+1}^{m} (1 + z_k) - 1 + 1 \]
\[ = \left( \prod_{k=n+1}^{m} (1 + z_k) - 1 \right) \]

So
\[ |p_m - p_n| = |p_n| \prod_{k=n+1}^{m} (1 + z_k) - 1 | \leq |q_n| \left( \prod_{k=n+1}^{m} (1 + z_k) - 1 \right) \]
\[ = q_m - q_n = 9_m - 9_n \]

Eq. 3 is convergent \( \iff \) Eq. 3 Cauchy \( \iff \) Eq. 3 is Cauchy \( \iff \) Eq. 3 is convergent.