Def: A series corresponding to a sequence \( \{ z_k \} \) is the sequence of partial sums \( \{ s_n \} \) where
\[
s_n = \sum_{k=1}^{n} z_k
\]
The series is often written as \( \sum z_k \).

\( \sum z_k \) is convergent if \( \{ s_n \} \) has a limit.

Def: A series \( \sum z_k \) is absolutely convergent if \( \sum |z_k| \) converges.

A convergent series that is not absolutely convergent is called conditionally convergent.
We will now launch into a discussion of monotonic sequences and Cauchy sequences.

Before we go there, note that the least upper bound property implies the greatest lower bound property. The greatest lower bound of $S$ is represented as $\inf S$.

**Proposition:** If a set $S$ is nonempty and bounded below, it has a greatest lower bound.

**Proof:**

$\underline{L} \leq S$

Construct $T = \{x | x \leq S \}$. It is nonempty and has an upper bound $-L$.

Thus, we have a least upper bound $t = \sup T$.

Now, $-t$ is going to be the greatest upper bound of $S$. 
Why? If there was a lower bound \( l \), s.t.

\[-t < l, \text{ but } x \geq l \text{ for all } x \in S, \text{ then} \]

\[-l \geq -x \quad \forall x \in S \Rightarrow -l \text{ is an upper bound of } S. \text{ Also } -t < l \Rightarrow t > -l \Rightarrow t \]

is not the least upper bound. Contradiction! QED

Now let us come to the first theorem concerning real sequences.

Thm: Every monotonic sequence is convergent if and only if it is bounded.

The 'only if' part is easy, since any convergent sequence is bounded. Let \( \{x_n\} \) have a limit \( x \). Then there is an \( N \), s.t.
\[ n > N \implies |x_n - x| < 1 \text{ (I chose } \varepsilon = 1 \text{ in our definition of convergent sequence.)} \]

So \( x-1 < x_n < x+1 \) for \( n > N \)

So the sequence is bounded above by \( \max\{x_1, \ldots, x_N, x+1\} \) and bounded below by \( \min\{x_1, \ldots, x_N, x-1\} \).

Now, the 'if' part:

Case a) \( \{x_n\} \) is increasing

b) \( \{x_n\} \) is decreasing

We will prove case a) using the least upper bound property. Case b) can be done analogously, by using the greatest lower bound property.
Consider the set $S = \{ x_n | n \in \mathbb{N} \}$.

If $S$ is bounded, it has an upper bound. So $\sup S$ exists. Set $x = \sup S$.

For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $x_N > x - \varepsilon$.

(Otherwise, $x$ is not the least upper bound.)

Therefore, $x - \varepsilon > x_n > x - \varepsilon$ for $n > N$.

Also, $x_n \leq x$.

So $\exists x_n$ converges to $x$.

Flip things around for decreasing sequence. Show that $\inf S$ would be the limit in that case. Q.E.D.
Let us pick an application. Consider the sequence \( \sum \left(1 + \frac{1}{n}\right)^n \). Is it convergent?

I am sure all of you know that this sequence converges to e. Sometimes it is used to define e.

Let us see if we can prove its convergence.

\[
x_n = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2)\cdots(1)}{n^n} \frac{1}{n^n}
\]

\[
= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n^n}
\]

Compare with \( \left(1 + \frac{1}{n+1}\right)^{n+1} \)

\[
x_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n+1}\right)^{n} \frac{1}{n^n}
\]

\[+ \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} \frac{1}{1 - \frac{1}{n+1}} \cdots \frac{1}{1 - \frac{1}{n+1}}
\]

The kth term has gotten bigger since \( (1 - \frac{1}{n}) < (1 - \frac{1}{n+1}) \) for \( k < n \). On top of that, \( x_{n+1} \) has an additional positive term.
So $x_{n+1} > x_n$

Also $x_n = 1 + \frac{1}{n} + \frac{\prod_{i=1}^{n-2} (1 - \frac{1}{i})}{n!} + \frac{1}{n!} (1 - \frac{1}{2}) (1 - \frac{1}{3}) \cdots (1 - \frac{1}{n})$

\[
\leq 1 + \frac{1}{2^1} + \frac{1}{3^1} + \cdots + \frac{1}{n^1} + \frac{1}{n!}
\]

Could I use it to prove $x_n$ is bounded above?

We will reuse it elsewhere, but

\[
1 + 1 + \frac{1}{2^1} + \frac{\prod_{i=2}^{n-2} (1 - \frac{1}{i})}{n!} + \frac{1}{n!} (1 - \frac{1}{2}) (1 - \frac{1}{3}) \cdots (1 - \frac{1}{n})
\]

is bounded above and could be used to prove that the series \(\sum_{k=1}^{\infty} \frac{1}{k^i}\) (another definition of \(e\)) is also convergent.

\[
1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots + \frac{1}{1.2.3 \cdots n}
\]

\[
< 1 + 1 + \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2.2.2} + \cdots + \frac{1}{2^{n-1}}
\]
= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 + 2 \left( 1 - \frac{1}{2^n} \right) < 1 + 2 = 3

So \( x_n < 3 \) for all \( n \). Also \( x_n \geq 2 \)

Since \( \{x_n\} \) is monotonic and bounded, it must have a limit.

Note that we also ended up proving \( \sum_{k=1}^{\infty} \frac{1}{k} \) is convergent, since partial sums keep increasing and the sum is always bounded above.

Our proof already shows \( \lim_{n \to \infty} (1 + \frac{1}{n})^n \leq e \).

What about the other way around? Turns out that

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e
\]

\[
x_n = 1 + 1 + (1 - \frac{1}{2}) \frac{1}{2} + \cdots + (1 - \frac{1}{k}) \frac{1}{k} + \cdots + (1 - \frac{1}{n}) \frac{1}{n}
\]

Stop at a fixed \( k \).

For \( n \geq k \)

\[
x_n \geq 1 + 1 + (1 - \frac{1}{2}) \frac{1}{2} + \cdots + 1 \cdot \left( 1 - \frac{1}{k} \right) \frac{1}{k}
\]

By taking \( n \to \infty \), \( x \geq 1 + 1 + \cdots + \frac{1}{k} \). This relieves...
Theorems on limits of sums and products, which I am assuming that you know. We also need that if two increasing convergent sequences \( \{a_n\}, \{b_n\} \) satisfy \( a_n \leq b_n \) \( \forall n \in \mathbb{N} \), then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

The last bit could have been helped by the concepts of \( \lim \inf \) and \( \lim \sup \). They also help us prove statements about real canceling sequences. They also would be useful for statements of root test and ratio test.

**Def.** Let \( \{x_n\} \) be a real sequence.

Define \( u_n = \sup \{x_m | m > n\} \)

\( l_n = \inf \{x_m | m > n\} \)

Note that \( \{u_n\} \) is decreasing, while \( \{l_n\} \) is increasing.
For monotonic sequences, we know the bounded ones converge. If they are not bounded, increasing ones go to $+\infty$ and decreasing ones go to $-\infty$. If we allow $\pm \infty$ as possible limits, the limit of a monotonic sequence is always defined.

Finally, we define $\limsup$ and $\liminf$:

$$\limsup x_n = \lim_{n \to \infty} \sup x_n$$

$$\liminf x_n = \lim_{n \to \infty} \inf x_n$$

By construction $\liminf x_n \leq \limsup x_n$.

For convergent sequences, these two are the same!