The Metric Tensor

Def. A smooth n-dimensional manifold \( M \) endowed with a smooth \((0,2)\) non-degenerate symmetric tensor field \( g \) is called a pseudo-Riemannian manifold. The tensor field \( g \) is called the metric tensor.

At any point \( p \in M \) if \( u, v \in T_p M \) we can compute a number \( g(p, u, v) \) which is a scalar product. In components, it looks like \( g_{ij} v^i v^j \). Note that

\[ u \rightarrow g_p(u, v) : \bar{v} \text{ is a map from } T_p M \text{ to } T^*_p M \]

In components \( u^i \rightarrow \bar{v}^i = g_{ij} u^j \) (note lowering).

Non-degenerate \( g \) means this map is invertible.

We have \( \bar{g}_p : T^*_p M \rightarrow T_p M \)

\[ \bar{g}^{jk} g_{kl} = \delta^j_k \quad \Rightarrow \quad \omega \rightarrow \bar{g}(\omega, \omega) \text{non-degenerate \( \Rightarrow \) \( \omega \), } \omega \bar{g} \text{ is invertible} \]

\( g \) is a symmetric matrix with non-zero eigenvalues. If it has \( p \) positive and \( q \) negative eigenvalues, then \( \text{signature of } g \) is \((p, q)\).
Metric Tensor and Distance

In Relativity theory, we deal with signatures like (1, 3). However, in this section, we focus on (0, 0) signature.

Def: If $g(u, u) \geq 0$ and $g(u, u) = 0$ implies $u = 0$, then $g$ is positive definite metric. A smooth manifold with a smooth positive definite metric is a smooth Riemannian manifold.

Now imagine you have a smooth curve on the manifold $\gamma: I \to M$ with $I = [a, b]$. At every point on the curve, we have a tangent vector. At point $\gamma(t)$, for $t \in I$, there is a vector $\gamma'(t) \in T_{\gamma(t)} M$. For this curve,

\[ l(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \, dt \]

We defined its length. If we describe the curve by $x^i = x^i(t)$,

\[ l(\gamma) = \int_a^b \sqrt{g_{ij}(\gamma'(t)) \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt \]
In textbooks talking about infinitesimal arc lengths, we say
\[ ds^2 = g_{ij} \, dx^i dx^j. \]

This is a rather useful tool for building intuition and we will gladly use it.

We could define \( s(a) = \int_a^1 \sqrt{g(\vec{x}(u), \vec{x}(u))} \, du \),
along the curve and show
\[ \left( \frac{ds}{du} \right)^2 = g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}, \]
which is a rigorous statement. This statement, depends on a parametrization, though.

Example: 1) Two-dimensional plane, \( \mathbb{R}^2 \):
\[ ds^2 = dx^2 + dy^2 \quad \text{Cartesian} \]
\[ = dr^2 + r^2 d\theta^2 \quad \text{Polar} \]

2) \( \mathbb{R}^3 \):
\[ ds^2 = dx^2 + dy^2 + dz^2 \]
\[ = dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

3) \( S^2 \):
\[ ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \]
Unit sphere
Metric Tensor and a Volume Form.

When we change coordinates:

\[ g_{jk} \text{ transforms to } g'_{mn} = \frac{\partial x^i}{\partial x^m} \frac{\partial x^k}{\partial x^n} g_{jk} \]

\[ J^m_n = \frac{\partial x^m}{\partial x^n} \]

So \( \det(g') = \det(g) \left[ \det(J) \right]^{-2} \)

Note that \( dx^1 \wedge \ldots \wedge dx^n = \frac{\partial x^1}{\partial x^i} \ldots \frac{\partial x^n}{\partial x^n} dx^1 \wedge \ldots \wedge dx^n \)

\[ = \det(J) \; dx^1 \wedge \ldots \wedge dx^n \]

So \( \sqrt{\det(g)} \; dx^1 \wedge \ldots \wedge dx^n \) is invariant under transformation.

For pseudo Riemannian metrics one often defines \( \sqrt{\det(g)} \; dx^1 \wedge \ldots \wedge dx^n \) as the volume element.

For the same reason anywhere in a connected part of \( M \).
The Laplacian operator

\[ \nabla^{2}(x) = \nabla(x) \cdot \nabla = -\Delta x^{n} \]

**g**

Now consider a contravariant vector associated with gradient of a function: \( \mathbf{\nabla} f = \mathbf{\nabla} f \cdot \mathbf{f} \)

In components, \( (\mathbf{\nabla} f)^{y} = g^{jk} \frac{\partial f}{\partial x^{k}} \)

* \( \mathbf{\nabla} f \in \Lambda^{n-1}(\mathbf{M}) \),
* \( d \mathbf{\nabla} f \in \Lambda^{n}(\mathbf{M}) \)

\( d \mathbf{\nabla} f \) is an \( n \)-form, \( \star d \mathbf{\nabla} f \) is back to \( \Lambda^{0}(\mathbf{M}) \)

\( \mathbf{\Delta} f = \sum_{j=1}^{n} \frac{\partial^{2} f}{\partial x^{j}} \)

\[ \begin{align*}
\Rightarrow \quad \mathbf{\Delta} f &= \sum_{j=1}^{n} \frac{1}{\sqrt{\text{det}(g)}} \frac{\partial}{\partial x^{j}} \left( \sqrt{\text{det}(g)} g^{jk} \frac{\partial f}{\partial x^{k}} \right) \\
&= \frac{1}{\sqrt{\text{det}(g)}} \frac{\partial}{\partial x^{j}} \left( \sqrt{\text{det}(g)} g^{jk} \frac{\partial f}{\partial x^{k}} \right)
\end{align*} \]

So we define, \( \mathbf{\Delta} f = \star d \mathbf{\nabla} f \) to be the Laplacian

Example: \( \mathbf{\Delta} s^{2} = d\mathbf{l}^{2} + 2 \mathbf{\dot{r}} \mathbf{\cdot} \mathbf{\dot{r}} + r^{2} \sin^{2} \mathbf{\theta} \)

\( \text{det}(g) = r^{4} \sin^{2} \mathbf{\theta} \)
\[ \Delta = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} r^2 \sin \theta \frac{2}{r} + \frac{\partial}{\partial \theta} r^2 \sin \theta \frac{2}{r^2 \theta} + \frac{\partial}{\partial \phi} r^2 \sin \theta \frac{2}{r^2 \sin \theta} \frac{2}{\partial \phi} \right] \]

Since \[ \bar{g} = \left( \begin{array}{c} 1 \\ \frac{1}{r^2} \frac{1}{\sin \theta} \end{array} \right) \]

\[ \Delta = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} r^2 \frac{2}{r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{2}{\partial \theta} \]

\[ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

Of course, in the Cartesian system:

\[ \bar{g} = \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \quad \bar{s} = \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \quad \text{So} \]

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
Geodesic curve on a manifold

\[ l = \int_a^b \sqrt{\sum_{j<k} g_{jk} \frac{dx^j}{dx^k} \frac{dx^j}{dx^k}} \, dx = \int_a^b \sqrt{\sigma} \, dx \]

\[ \eta^i(a) = \eta^i(b) = 0 \]

\[ \frac{d}{dt} \left[ \frac{1}{2} \sum_{j<k} g_{jk} \eta^j \eta^k \right] = \frac{d}{dt} \left[ \sum_{j<k} g_{jk} \eta^j \frac{d\eta^k}{dt} \right] + \sum_{j<k} g_{jk} \frac{d\eta^j}{dt} \frac{d\eta^k}{dt} \]

\[ = \frac{1}{2} \int_a^b \left[ \sum_{j<k} \frac{d}{dt} \left( g_{jk} \eta^j \frac{d\eta^k}{dt} \right) + \frac{d}{dt} \left( \sum_{j<k} g_{jk} \frac{d\eta^j}{dt} \frac{d\eta^k}{dt} \right) \right] \, dx \]

\[ = \int_a^b \left[ \frac{1}{2} \frac{d}{dt} \left( \sum_{j<k} g_{jk} \frac{d\eta^j}{dt} \frac{d\eta^k}{dt} \right) - \frac{1}{2} \frac{d}{dt} \left( \sum_{j<k} g_{jk} \frac{d\eta^j}{dt} \frac{d\eta^k}{dt} \right) \right] \eta^m \, dx \]

\[ = \int_a^b \left[ \frac{1}{2} \frac{d}{dt} \sum_{j<k} g_{jk} \frac{d\eta^j}{dt} \frac{d\eta^k}{dt} \right] \eta^m \, dx \]

Read and Understood By
We can plug in $\sigma = g_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda}$ and push on (the book does). We should realize $L(\gamma)$ does not depend on the parametrization.

If we go from $x^i(\lambda)$ to $x^i(\lambda') = \tilde{x}^i(\mu)$, this will also be a solution. To fix this degeneracy, we define $ds$ to be $\sqrt{g}$.

So $S$ is the arc length.

$$\frac{dl}{d\lambda} = -\int \left[ \frac{d}{ds} \left( g_{jm} \frac{dx^j}{ds} \right) - \frac{1}{2} \frac{d}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{1}{\sqrt{g}} \gamma_{jm} \gamma_{kn} \right] \sqrt{g} d\lambda$$

Since $\gamma_{jm}$ is arbitrary,

$$\frac{d}{ds} \left( g_{jm} \frac{dx^j}{ds} \right) - \frac{1}{2} \frac{d}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

$$g_{jm} \frac{d^2 x^j}{ds^2} + \left( \frac{d}{ds} \left( g_{jm} \frac{dx^j}{ds} \right) - \frac{1}{2} \frac{d}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{1}{\sqrt{g}} \gamma_{jm} \gamma_{kn} \right) \frac{d}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

$$g_{jm} \frac{d^2 x^j}{ds^2} + \Gamma_{jm} \frac{dx^i}{ds} \frac{dx^k}{ds} = 0$$
\[ \Gamma_{mijk} \] are Christoffel symbols (of the 1st kind).

\[ \Gamma_{mijk} = \frac{1}{2} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) \]

Use \( g^{im} \) to raise indices with \( \Gamma^i_{jk} = g^{im} \Gamma_{mijk} \) (Christoffel Symbols of the 2nd kind).

\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0. \]

We have the Christ  \( g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = 1 \) satisfied.

\( \Gamma^i_{jk} \) is associated with connections and parallel transport, allowing comparison of vectors in neighboring tangent spaces.

Example: \( S^2 \):

\[ g_{\theta \theta} = 1, \quad g_{\phi \phi} = \sin^2 \Theta \]

Only nontrivial derivative:

\[ \frac{\partial g_{\phi \phi}}{\partial \Theta} = 2 \sin \Theta \cos \Theta \]

So

\[ \Gamma^\phi_{\Theta \phi \Theta} = \frac{1}{2} \left( \frac{\partial g_{\phi \phi}}{\partial \Theta} \right) = -\sin \Theta \cos \Theta \]

\[ \Gamma^\phi_{\Theta \phi \Theta} = \Gamma^\phi_{\Theta \phi \Theta} = \frac{1}{2} \left( \frac{\partial g_{\phi \phi}}{\partial \Theta} \right) = \sin \Theta \cos \Theta \]

\[ \Gamma^\phi_{\Theta \phi \Theta} = \Gamma^\phi_{\Theta \phi \Theta} = \frac{1}{2} \left( \frac{\partial g_{\phi \phi}}{\partial \Theta} \right) = \sin \Theta \cos \Theta \]

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So the equations are
\[ \frac{d\theta}{ds^2} + R \frac{d}{ds} \left( \frac{d\phi}{ds} \right)^2 = 0, \quad \text{and} \quad \frac{d\phi}{ds^2} + 2 \frac{d}{ds} \frac{d\phi}{ds} \frac{d\theta}{ds} = 0. \]

Or
\[ \dot{\theta} - \sin \phi \cos \theta \dot{\phi}^2 = 0 \quad \text{and} \quad \dot{\phi} + 2 \cos \phi \dot{\theta} \dot{\phi} = 0. \]

Note that if we have \( \phi = 0 \) initially, then \( \theta = \omega \) is a solution.

The solution moves along \( \phi = \) constant lines providing great circles. Since \( \frac{d\theta}{ds} = 1 \)

\[ \omega = \pm 1. \]

For general \( \phi \), recast second equation as \( \frac{d}{ds} (\phi \sin^2 \theta) = 0 \)

or \( \frac{d\theta}{ds} = \frac{c}{\sin \theta} \left( \text{"conserved quantity"} \right) = \frac{c^2 \cos \theta}{\sin \theta} = 0 \)

implying

\[ \frac{1}{2} \dot{\theta}^2 + \frac{c^2}{2 \sin^2 \theta} = \frac{1}{2} A \quad \text{(like energy)} \]

\( \text{Note that we have} \quad \dot{\theta}^2 + \sin^2 \dot{\phi}^2 = A, \text{which has to be 1.} \)

So \( \dot{\theta}^2 + \frac{c^2}{\sin^2 \theta} = 1 \) and \( \dot{\phi} = \frac{c}{\sin \theta} \quad \rightarrow \quad \frac{d\phi}{d\theta} = \frac{c}{\cos \theta} \quad \rightarrow \quad \frac{d\phi}{d\theta} = \frac{d\phi}{d\theta} \quad \rightarrow \quad \int \frac{d\phi}{\sqrt{1 - c^2 \sin^2 \phi}} = \frac{c}{\sin \theta} \)

\( \int d\phi = \int \frac{c \, du}{\sqrt{1 - c^2 - c^2 u^2}} = \sqrt{\frac{\sin \theta}{\sin \theta}} \quad \rightarrow \quad \int d\phi = \sin^{-1} (c \sin \phi) \quad \rightarrow \quad \phi = \sin^{-1} (c \sin \phi) \)

\( \cos \phi = \sin \phi \cos \theta \quad \text{and} \quad \sin \phi = \sqrt{1 - \sin^2 \phi} \quad \rightarrow \quad \cos \phi = \sqrt{1 - \sin^2 \phi} \cos \theta \quad \rightarrow \quad \cos \phi = \tan \theta \sin (\phi - \phi) \)

\( \sin (\theta - \phi_0) = \frac{\sin \phi \cos \theta}{\sqrt{\sin^2 \phi + \cos^2 \theta}} \quad \rightarrow \quad \cos \theta = \tan \theta \sin (\phi - \phi_0) \)

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