Wedge Product, Exterior/Grassmann Algebra

Consider tensors in $V \otimes \cdots \otimes V$. They can also be thought as multilinear maps from $V^* \times \cdots \times V^*$ to $\mathbb{R}$.

Let $T(u_1, \ldots, u_p) \rightarrow T(u_1, \ldots, u_p)$.

We can define subspaces of $V \otimes \cdots \otimes V = \otimes^p V$ by symmetry. Say $T \in S_p$ by permutation.

$T(u_{(1)}, \ldots, u_{(p)}) = T(u_1, \ldots, u_p)$ defines symmetric tensors,

while

$T(u_{1(1)}, \ldots, u_{1(p)}) = \sum_{\sigma \in S_p} \text{sgn}(\sigma) T(u_1, \ldots, u_p)$ defines antisymmetric tensors.

The second subspace, consisting of antisymmetric tensors, is indicated as $\Lambda^p V$. Notice that it is $(\binom{p}{2})$-dimensional space where $n = \dim V$. 

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We define a linear transformation $\mathbf{A}$ in an example of the bulk product

\[ A = \frac{1}{2} \mathbf{a} \otimes T - \mathbf{a} \otimes T \]

Consider these two combinations of their tensor products.

Let us start with two forms $\omega, \eta$. We will essentially focus on antisymmetric covariance tensors.

Although the construction in general, $V$ would be $T^*M$. We will essentially focus on antisymmetric covariance tensors.
Let us give this definition a trial run.

What is \( 0, \Lambda \sigma_2 \Lambda \sigma_3 \)?

\[
\begin{align*}
\text{even permutations: } & (1,2,3), (2,3,1), (3,1,2) \\
\text{odd permutations: } & (2,1,3), (3,2,1), (1,3,2)
\end{align*}
\]

So \( 0, \Lambda \sigma_2 \Lambda \sigma_3 = \frac{1}{6} \left[ \sigma_1 \otimes \sigma_2 \otimes \sigma_3 + \sigma_2 \otimes \sigma_3 \otimes \sigma_1 + \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \\
- \sigma_2 \otimes \sigma_1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_2 \otimes \sigma_1 - \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \right] \)

If \( \mu^i \) form some basis for the vector space \( V \), the general member of \( \Lambda^r V \) can be written as \( \sigma = \frac{1}{r!} \sigma^{i_1 \cdots i_r} \mu_{i_1} \wedge \cdots \wedge \mu_{i_r} \)

(we have summation convention, remember!)

If \( \sigma \in \Lambda^p V \) and \( T \in \Lambda^q V \)

\( \sigma \wedge T = \frac{1}{p! \cdot q!} \sigma^{i_1 \cdots i_p} T_{i_1 \cdots i_p} \wedge \cdots \wedge \mu_{i_1} \wedge \cdots \wedge \mu_{i_r} \)

We have to check it is consistent with our definition for one form and has associativity.
\[ 0 \wedge T = (-1)^p T \wedge 0 \]

With the wedge product we have full algebra

\[ \Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V) \], called the exterior algebra or Grassmann algebra.

Notice that \( \Lambda^n(V) \) is one-dimensional with the basis element \( \mu^i \wedge \ldots \wedge \mu^m \). If we change basis to \( \xi^i \) with \( \mu^i = s^i_j \xi^j \)

\[ \mu^i \wedge \ldots \wedge \mu^m = s^i_1 \ldots s^i_m \xi^1 \wedge \ldots \wedge \xi^m \]

\[ = \varepsilon^{i_1 \ldots i_m} s^i_1 \ldots s^i_m \xi^1 \wedge \ldots \wedge \xi^m = \det(s) \Lambda^i \wedge \ldots \wedge \Lambda^m \]

This property would be very important for differential forms and integration.

In vector calculus in 3D, you have seen line integral \( \int \vec{A} \cdot d\vec{r} \)

surface integral \( \int \vec{B} \cdot d\vec{S} \)

and volume integral \( \int \vec{C} \cdot d\vec{V} \)

These are all related to forms.
Before we move into integrals, let us define one more thing, the Hodge * operator.

Let us say we have a non-zero \( n \)-form \( \omega \) in \( \Lambda^n(V) \). If \( \sigma \in \Lambda^p(V) \) and \( \tau \in \Lambda^{n-p}(V) \), then \( \sigma \wedge \tau \in \Lambda^n(V) \).

Since \( \Lambda^n(V) \) is one-dimensional,

\[
\sigma \wedge \tau = c(\sigma, \tau) \omega
\]

where \( \sigma \mapsto c(\sigma, \tau) \) is a linear map on \( \Lambda^{n-p}(V)^* \), which lives in \( \Lambda^p(V^*) \). If \( V \) has an inner product, one could define a map from \( \Lambda^{n-p}(V^*) \to \Lambda^{n-p}(V) \), and \( c(\sigma, \tau) = \langle \sigma, \tau \rangle \).

Let us see that in an example:

Take \( A = A_1 e^1 + A_2 e^2 + A_3 e^3 \) as 1-forms,

\( B = B_1 e^1 + B_2 e^2 + B_3 e^3 \)

Then

\[
A \wedge B = (A_1 B_2 - A_2 B_1) e^1 \wedge e^2 + (A_2 B_3 - A_3 B_2) e^2 \wedge e^3 + (A_3 B_1 - A_1 B_3) e^3 \wedge e^1
\]

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This is a two form whose components look like it came from the familiar cross product.

Now consider $C = c_1 e^1 + c_2 e^2 + c_3 e^3$

\[ (A \wedge B) \wedge C = (A_1 B_2 - A_2 B_1) C_3 (e^1 \wedge e^2) \wedge e^3 + (A_2 B_3 - A_3 B_2) C_1 (e^2 \wedge e^3) \wedge e^1 + (A_3 B_1 - A_1 B_3) C_2 (e^3 \wedge e^1) \wedge e^2 \]

Choose scalar product as the usual dot product.

Then \[ (A_2 A_3 - A_3 A_2) e_1 + (A_3 A_1 - A_1 A_3) e_1 \]

\[ + (A_1 A_2 - A_2 A_1) e_2 \]

in our Hodge $* \left( A \wedge B \right)$

So, cross product in the Hodge dual in $3d$ of the two form created by wedge product.

That is why, it is pseudovector. In fact, many pseudovectors you know are really 2-forms.
In coordinates, if \( J^2(x) = f(x) \), we have:

\[
\sigma \in \Lambda^p(T_x^* M) \rightarrow v \in \Lambda^p(T_x M)
\]

\[
\nabla_{i_1 \ldots i_n} = \frac{1}{p!} \frac{1}{f(x)} \varepsilon_{i_1 \ldots i_n} \nabla_{i_1 \ldots i_n} \delta_{i_1 \ldots i_n}^{i_1 \ldots i_n}
\]

To make a covariant tensor out of \( \nabla_{i_1 \ldots i_n} \), we need a metric tensor \( g_{ij} \) to lower indices. Such a metric is also associated with a volume form. More about it later!

For completeness sake, from any antisymmetric and contravariant tensor, we can get a (\( r \)) antisymmetric covariant tensor:

\[
\Omega_{i_1 \ldots i} = \frac{1}{(n-r)!} P(x) \delta_{i_1 \ldots i} \nabla_{i_1 \ldots i} \delta_{i_1 \ldots i}
\]
Let us now think of integrals.

Imagine there is an open set $U \subset \mathbb{C}^n$, $n$-dimensional manifold, and have an $n$-form field $\omega$ defined on $U$. Let us also have a chart with $\varphi: U \to \mathbb{R}^n$. $\varphi$ is well defined.

Consider the pullback $(\varphi^{-1})^* \omega$.

That is $f(x) \, dz^1 \ldots dz^n$.

\[
\int_U \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega = \int f(x) \, dx^1 \ldots dx^n
\]

thought of as a higher dimensional Riemann integral. Technically we need $\omega$ to have compact support, so $f$ has compact support and we can find a rectangle in $\mathbb{R}^n$ which includes the support of $f$.

Notice that this is a signed integral $\int f \, dx_1 \ldots dx_n$.

For convenience we define $\int_{\partial U} \omega = \int_{\partial \Omega} 1 \, dx_1 \ldots dx^n$.

The conventional Riemann integral.
Now imagine we have two charts

\[ S_{uv} \]

\[ \varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ \psi: \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ S_{uv} = \int (\varphi^{-1})^* w \quad \text{and} \quad \int (\psi^{-1})^* w \]

\[ \varphi(U_{uv}) \]

Let the corresponding points \((x^1, \ldots, x^n)\) and \((x'^1, \ldots, x'^n)\).

\[ f(x) \quad dx^1 \wedge \cdots \wedge dx^n \]

transforms to

\[ f(x) \quad \frac{\partial x^1}{\partial x'^1} \cdots \frac{\partial x^n}{\partial x'^n} \quad dx'^1 \wedge \cdots \wedge dx'^n \]

\[ = f(x) \quad \det \left| \frac{\partial x^i}{\partial x'^j} \right| \quad dx'^1 \wedge \cdots \wedge dx'^n \]

\[ = h(x') \quad dx'^1 \wedge \cdots \wedge dx'^n \]

We want \( \int f(x) \quad dx^1 \cdots dx^n \) and \( \int h(x') \quad dx'^1 \cdots dx'^n \)

to be the same.
If we define them as Riemann integrals with integrations in the increasing direction, then we need more.

One solution is to have charts so that $\det\left( \frac{\partial x^i}{\partial x'^j} \right)$ is always positive! This leads to an oriented atlas.

If it is possible to have such a chart, we will call the manifold orientable.

Cannot be done for manifolds like a Möbius strip.
Technically, to define integration on a manifold, we need to deal with a patchwork of charts \( \phi_{\alpha}, \phi_{\beta} \).

One can show that one can find a set of functions \( \phi_i \), taking values in \([0,1]\) and such that \( \phi_i \) is supported on \( U_2 \), namely \( \text{supp}(\phi_i) \subset U_2 \) for some \( i \).

Make \( \phi_i \) a function. Also at each \( p \in M \) only a finite number of \( \phi_i \)'s are non-zero. In addition

\[
\sum \phi_i(p) = 1 \quad \text{at each } p.
\]

These are called a partition of unity.

Then integral

\[
\int_M \omega = \int_M \sum_i \phi_i \omega = \sum_i \phi_i \omega 
\]

Each of these integrals are well-defined.
Now, to do integrals inside a manifold, we need a few more concepts.

One is the concept of manifolds with boundaries. We need to generalize charts mapping to $\mathbb{H}^n$ from $\mathbb{R}^n$

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x^n > 0\}$$

Second thing is that the points that map to a point with $x^n = 0$ are on the boundary and have a natural chart with $(x', \ldots, x^{n-1}) \Rightarrow$ Induced atlas of the manifold of Call these points $\partial M$.

Turns out that if $M$ is oriented, the induced atlas is oriented.

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Consider now a submanifold \( R \) of \( M \). \( R \) could potentially have a boundary \( \partial R \). Let the inclusion map \( i: R \to M \) be smooth. For any form \( \omega \) on \( M \), there is pullback \( i^* \omega \) on \( R \).

If the dimension of \( R \) is \( p \) and \( \omega \) is a \( p \)-form with compact support on \( R \), we define

\[
\int_{i^* \omega} \to \int_R \omega.
\]
Let us see this in action. Imagine that there is a 2D surface in 3D. \((u, v)\) in the local coordinate system on the surface and \(x(u, v), y(u, v), z(u, v)\) provides a parametrization of the surface. Let there be a two form \(B = B(r) dy dz + B_s dx dy\) in 3D. In vector calculus, you will write it as \(\bar{B} \cdot dS\). \((B_1, B_2, B_3)\) being the Hodge dual of \(B\).

The integral is

\[
\int Byz \frac{\partial y}{\partial (u,v)} \, du \, dv \\
+ \int Bzx \frac{\partial z}{\partial (u,v)} \, du \, dv \\
+ \int Bxy \frac{\partial x}{\partial (u,v)} \, du \, dv
\]