Vector Fields and Integral Curves

Let \( \mathbf{v} \) be a vector field on a manifold \( M \).

Let \( \gamma: I \rightarrow M \) be a curve (\( I \) being an interval in \( \mathbb{R} \)).

This curve gives rise to a tangent at any point it passes through via the \( \frac{d}{dt} \) operator

\[
\frac{d}{dt} f(x(t)) = \frac{df}{dx} \quad \text{for any smooth } f: M \rightarrow \mathbb{R}
\]

If \( \frac{d}{dt} = \mathbf{v}(x(t)) \), then \( \gamma \) is an integral curve of the vector field \( \mathbf{v} \).

More explicitly, in coordinates,

\[
\frac{d}{dt} x_i(x(t)) = v^i(x(t))
\]

or

\[
\frac{d}{dt} x_i = v^i(x) \quad \text{in short}
\]
Note that these are a set of differential equations. If $\psi(t)$ are continuous (and Lipschitz continuous), they are going to have a unique local solution.

The map we get by starting out at a point with coordinates $x^i = x^i(0)$ and solve the equations $x^i(t)$. This solution creates a map starting at point $p$ with coordinates $x^i$ and maps it to a new point that depends on $x$ and $\psi$. We call this map $\psi(p, \lambda): M \rightarrow M$.

This turns out to be a smooth invertible map from $M$ to $M$. To discuss it further, we need to understand smooth maps between smooth manifolds.
Manifolds and smooth mappings between them.

Let $F : M \to N$

Def: $F$ is smooth if $F \circ \phi^{-1}$ is smooth in a neighborhood of $\phi(p)$, for every $p$, and any chart $\phi$, including $p$ and $F(p)$, respectively.

At any point $p \in M$, we can define a pushforward of a vector $v_p$ from $T_p M \to T_{F(p)} N$.

Def: The linear map $F_* : T_p M \to T_{F(p)} N$ defined by $F_* v_p(f) = v_{F(p)}(f \circ F)$, for any smooth $f : N \to \mathbb{R}$. 

Read and Understood By
In coordinates, with slightly sloppy notation,

\[ f_0^* (F) = \begin{pmatrix} \frac{\partial y_j}{\partial x^i} \end{pmatrix} \]

\[ (F_0^* V_p) = \frac{\partial y^j}{\partial x^i} V_0^i = P^j V^i \]

Note, by the way, we cannot always have push-forward.

If \( F \) is a many-to-one map, we do not have a well-defined field on \( N \).

The push-forward of vector fields makes sense if \( F \) is a diffeomorphism.
Pullback of a 1-form

$$\omega_p \in T_{F(p)}^* N$$

**Def:** The pullback $$F^* \omega_p \in T_p^* M$$ is defined by $(F^* \omega_p)(\nu_p) = \omega_p(F^*(\nu_p))$

for any $\nu_p \in T_{F(p)}^* M$.

Notice that we could define pullback of smooth 1-form fields from $N$ to $M$. Contrast this with the requirement for push-forward of vector fields.
In coordinates

\[ (w \cdot dy_i) \rightarrow (F^* w) \cdot dx_i = \omega_j \frac{\partial y_j}{\partial x_i} \cdot dx_i \]

\[ \omega_j (y(x), \ldots, y^n(x)) \frac{\partial y_j}{\partial x_i} \] would be the pulled back smooth form on \( M \).

Note that this is \( \omega_j F^i \).

In matrix language \((F^* \omega) = \frac{\omega}{\omega} \cdot \text{column vector} \)

\[ (F^* \omega) = \omega \cdot F \]

\( F \in \mathbb{R}^{n \times m} \)

\( v \in \mathbb{R}^{m \times 1} \)

\( \omega \in \mathbb{R}^{1 \times n} \)

By the way, any covariant tensor field could be pulled back:

\[ (F^* T)(v_1, \ldots, v_s) = T(F^* v_1, \ldots, F^* v_s) \]

Using this local pullback, we can use this to define fields everywhere. In components

\[ T_{i_1 \ldots i_s} \quad \frac{\partial y_1}{\partial x_{i_1}} \ldots \frac{\partial y_s}{\partial x_{i_s}} \]
Now, let us go back to the map created by a vector field, namely \( \Phi^r_0(\vec{x}) \). The map
\[
F_r(\vec{x})
\]
is a differomorphism from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

We can define push-forward of vectors \( \mathbb{R}^n \rightarrow F_\ast \mathbb{R}^n \) and pull back of forms \( \mathbb{R}^n \rightarrow F_\dagger \mathbb{R}^n \). However, if the local linear maps a full rank, we can also invert them.

Consider a small transformation:

\[
F_r(\vec{p}, \varepsilon) \xrightarrow{\Phi} (x_1^r + \varepsilon \frac{\partial}{\partial x_1} x^r_1, \ldots, x^n_1 + \varepsilon \frac{\partial}{\partial x_1} x^r_n) + o(\varepsilon^2)
\]

Let us use the same coordinates. So
\[
y^i = x^i + \varepsilon \frac{\partial}{\partial x_1} x^r_i + o(\varepsilon^2)
\]

\[
F_r^j_i = \frac{\partial y^j}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ x^j_i + \varepsilon \frac{\partial}{\partial x_1} x^r_j + o(\varepsilon^2) \right] = \delta^j_i + \varepsilon \frac{\partial}{\partial x_1} x^r_j + o(\varepsilon^2)
\]

\[
e_i = \frac{\partial}{\partial x_i} \rightarrow F(e_i) = \left( \delta^j_i + \varepsilon \frac{\partial}{\partial x_1} x^r_j \right) e_j + o(\varepsilon^2)
\]
So \( F^*e_i = e_i + \epsilon \frac{\partial}{\partial x^i} e_j = \mathcal{O}(\epsilon^2) \)

For pullback of one forms: \( F^* = (\delta^i_j + \epsilon \frac{\partial u_i}{\partial x^j} \epsilon + \mathcal{O}(\epsilon^2)) \)

If \( F^* \) is invertible

\[ F^* e_i = \]

\[ = \left( \delta^i_j - \epsilon \frac{\partial u_i}{\partial x^j} \right) e^j + \mathcal{O}(\epsilon^2) \]

\[ = e^i - \epsilon \frac{\partial u_i}{\partial x^j} e^j + \mathcal{O}(\epsilon^2) \]

Note that

\[ \langle F^* e_i, F^* e_j \rangle = \]

\[ = \left( e^i - \epsilon \frac{\partial u_i}{\partial x^j} e^j \right) \left( e^j + \epsilon \frac{\partial u_j}{\partial x^i} e^i \right) + \mathcal{O}(\epsilon^2) \]

\[ = \langle e^i, e^j \rangle + \epsilon \frac{\partial u_i}{\partial x^j} e^j + \mathcal{O}(\epsilon^2) \]
\[ \delta_j^i + \varepsilon \sum_{k} \frac{\partial u_i}{\partial x^k} \delta_k^j - \varepsilon \sum_{k} \frac{\partial u_i}{\partial x^j} \delta_k^k + o(\varepsilon^2) \]

Note the two leading order "push-forward" of the dual basis remain a dual basis.

So, using \( F_p(x) \), we can push forward a tensor from \( T^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_m}(p) \) to \( T^{\sigma_1\sigma_2\cdots\sigma_n}_{\tau_1\tau_2\cdots\tau_m}(F(p)) \).

One way is to write the tensor as

\[ T^{i_j}_{i_{j_1}}(x, e) = e_i \otimes \cdots \otimes e_i \otimes \cdots \otimes e_i \text{ at } p \]

and map it to

\[ T^{i_j}_{i_{j_1}}(x) = e_{i_{j_1}} \otimes \cdots \otimes e_{i_{j_1}} \otimes \cdots \otimes e_{i_{j_1}} \]

and compare it to

\[ T^{i_j}_{i_{j_1}}(x + \varepsilon v) = e_i \otimes \cdots \otimes e_i \otimes \cdots \otimes e_i \]

In other words, we would like to calculate the derivative

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ T^{i_j}_{i_{j_1}}(p + \varepsilon v) - T^{i_j}_{i_{j_1}}(p) \right] \]

Continued on Page...
This is called the Lie derivative \( \mathcal{L}_\omega \) w.r.t. the vector field \( \omega \).

Let us get some idea about what the Lie derivative does with some lower level tensors. We will represent by \( e_i(\mathbf{p}) \) and \( e^i(\mathbf{p}) \), the vector and the form bases at \( \mathbf{p} \).

1. \((0,0)\) order tensors or functions:

\[
\frac{\partial f}{\partial x^i} = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{v}) - f(\mathbf{x})}{\epsilon}
\]

\[
= v^i \frac{\partial f}{\partial x^i}
\]

It is just the directional derivative.

2. Vector fields:

If \( \mathbf{w} \) is a vector field, \( \mathbf{w} = \sum \mathbf{e}_i \partial f/\partial x^i \), at \( \mathbf{p} \):

\[
\mathcal{L}_\omega \mathbf{w} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \mathbf{w}(\mathbf{x} + \epsilon \mathbf{v}) \cdot \mathbf{e}_i(\mathbf{p}) - \mathbf{w}(\mathbf{x}) \cdot \mathbf{e}_i(\mathbf{p}) \right]
\]

\[
= \omega^i(\mathbf{x}) \left( \delta \mathbf{w}/\delta x^i \cdot \mathbf{e}_j(\mathbf{p}) \right) - \omega^i(\mathbf{x}) \frac{\partial \mathbf{w}}{\partial x^i} \cdot \mathbf{e}_j(\mathbf{p})
\]

\[
= \omega^i(\mathbf{x}) \delta \mathbf{w}/\delta x^i \cdot \mathbf{e}_j(\mathbf{p}) - \omega^i(\mathbf{x}) \frac{\partial \mathbf{w}}{\partial x^i} \cdot \mathbf{e}_j(\mathbf{p})
\]

Continued on Page
Note that $\mathbf{L} \omega = - \mathbf{L} \mathbf{u} \cdot \mathbf{v}$. Antisymmetry.

In fact $\omega(f) = \mathbf{u} \cdot \frac{\partial}{\partial x^i} f$ is a function. If we apply $\mathbf{L}(\omega(f)) = \mathbf{u} \cdot \frac{\partial}{\partial x^i} \left( \mathbf{w} \cdot \frac{\partial f}{\partial x^i} \right)$

$= \left( \frac{\partial}{\partial x^i} \mathbf{u} \cdot \mathbf{v} \right) \frac{\partial f}{\partial x^i} + \mathbf{u} \cdot \mathbf{v} \frac{\partial^2 f}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial x^i}$

Note that this is not a derivation because of the second derivative term and hence this operator on functions is not a vector field.

On the other hand, $\mathbf{L}(\omega(f)) = \mathbf{u} \cdot \frac{\partial}{\partial x^i} \frac{\partial f}{\partial x^i}$

$= \left( \frac{\partial}{\partial x^i} \mathbf{u} \cdot \mathbf{v} \right) \frac{\partial f}{\partial x^i} + \mathbf{u} \cdot \mathbf{v} \frac{\partial^2 f}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial x^i}$

is a vector field's action on $f$.

So $\mathbf{L} \omega = [\mathbf{v}, \omega]$.

If $e_i = \frac{\partial}{\partial x^i}$ is a coordinate basis, what is $\mathbf{L} e_i$? Well $[e_i, e_j] = \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$.
What is the map $F_{e_i} : \mathbb{R}^n \to \mathbb{R}^n$?

\[
\frac{dx^i}{dt} = 1, \quad \frac{dx^j}{dt} = 0 \quad \text{for } j \neq i.
\]

So $(x^1, \ldots, x^i, \ldots, x^n) \mapsto (x^1, \ldots, x^i+\varepsilon, \ldots, x^n)$.

If we consider flows $F_{e_i} : (x, \varepsilon) \to (x, \varepsilon)$ and $F_{e_j} : (x, \varepsilon) \to (x, \varepsilon+\varepsilon_2)$, they commute.

\[
(x^1, \ldots, x^i, \ldots, x^n) \text{ just goes to } (x^1, \ldots, x^i+\varepsilon, \ldots, x^n+\varepsilon_2). \]

In general, it does not commute, the same for $e_j$.

Each coordinate $x^i$ thought of as a function:

\[
x^i : \mathbb{R} \to \mathbb{R}, \quad (x^i)_{\varepsilon_2} = x^i_{\varepsilon_2} : \mathbb{R} \to \mathbb{R}.
\]

This is the mismatch of coordinates.
If we need to check whether vector fields \( v_1, \ldots, v_n \) correspond to some \( \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \) for some coordinate system \( y^1, \ldots, y^n \), the necessary condition is \[ \sum_{i<j} v_i \wedge v_j = 0 \] for all pairs \( i, j \).

3. Moving on to 1-forms

\[ \omega = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \omega(x + \varepsilon y^i) - \omega(x) \right] e_i \]

\[ = \left( \frac{\partial \omega_i}{\partial x^j} + \omega_j \frac{\partial \omega_i}{\partial y^j} \right) e_i \]

4. Mixed Tensor

\[ \delta e T = \left( \frac{\partial k^2}{\partial x^2} T_{i, i} - T_{i, i} \right) \delta e_i \]