1  [8 pts] The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.
(a) [4 pts] Assuming that the required derivatives exist, show that
\[ \nabla^2 u = \nabla^2 v = 0. \]
(b) [4 pts] Show that
\[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} = 0 \]
and give a geometric interpretation.
[Hint: what are the normals to the curves $u = \text{constant}$ and $v = \text{constant}$?]

2  [5 pts] Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, y) + iv(x, y)$. We label the real part $u(x, y)$ the velocity potential and the imaginary part $v(x, y)$ the stream function. The fluid velocity $\vec{V}$ is given by $\vec{V} = \nabla u$. If $f(z)$ is analytic,
(a) Show that $df/dz = V_x - iV_y$,
(b) Show that $\nabla \cdot \vec{V} = 0$ (no sources or sinks),
(c) Show that $\nabla \times \vec{V} = 0$ (irrotational, nonturbulent flow).

3  [8 pts] (a) [4 pts] If $f(z)$ is analytic and bounded [$|f(z)| \leq M$ for some constant $M$] for all $z$, show that $f(z)$ is a constant. This is called Liouville’s theorem (complex analysis version. He also has others).
(b) [4 pts] Prove that every polynomial of order $n \geq 1$ with constant coefficients $\in \mathbb{C}$ has at least one root. This is the fundamental theorem of algebra.