1. Consider the unit sphere $S^2$. Embedded in three dimensional Euclidean space $\mathbb{R}^3$ with cartesian coordinates, this is the set $\{(x, y, z) | x^2 + y^2 + z^2 = 1\}$, but as it is a two dimensional space, we usually describe it with spherical polar coordinates $\theta, \phi$, fixing the radius $r = 1$, with $z = \cos \theta$ and $(x, y) = \sin \theta (\cos \phi, \sin \phi)$. The coordinates $\theta$ and $\phi$ are well-defined, however, only for $\theta \neq 0$ and $\neq \pi$. So we have a chart $C$ for $U$ the unit sphere with the north and south pole removed.

The unit sphere has a natural metric induced from the Euclidean metric of $\mathbb{R}^3$, with which we are very familiar:

$$(ds)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2,$$

or $g_{\theta\theta} = 1, g_{\phi\phi} = \sin^2 \theta, g_{\theta\phi} = 0$.

(a) Calculate the Christoffel symbol $\Gamma^j_{\ k\ell}$

(b) The shortest path (on the sphere) between two points is determined by the geodesic equations. Find the equations which determine $\theta(s)$ and $\phi(s)$, where the $s$ parameter is the path length. Show that one of these second order equations is essentially the conservation of $L_z$.

(c) It is easier to guess the solution from the knowledge that the answer is a great circle, the intersection of the sphere with a plane though the origin, rather than solving the equations from scratch. Show that this is correct, satisfying the geodesic equations.

2. Consider the unit sphere $S^2$ as the set of point $\{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$. This is really a two dimensional manifold, so it needs to be considered as such. Consider the two open sets $U_N$ which is the sphere minus the South pole, and $U_S$, the sphere minus the North pole. Define the charts $C_N$ and $C_S$ by considering the equatorial plane $\mathcal{E}$. For each point $P \in U_N$ on the sphere, draw straight lines (in $\mathbb{R}^3$) through the South pole and $P$, and define $\phi_N(P)$ to be the point $(x_N, y_N)$ at which the line intersects the equatorial plane. Similarly define $\phi_S(P)$ to be the point in equatorial plane on the straight line through the North pole and $P$, but call that point $(x_S, -y_S)$. [The reason for inverting $y$ is to keep $dx_S \wedge dy_S$ pointing to the outside of the sphere.]

(a) Find the coordinates of $\phi_N(P)$ in terms of the $(x, y, z)$ of $P$ in $\mathbb{R}^3$, and the inverse relation.

(b) Do the same for $\phi_S(P)$, and find the transition function

$$\phi_{SN} = \phi_S \circ \phi_N^{-1} = \phi_S (\phi_N^{-1}(x_N, y_N))$$.

(c) The metric on the sphere is $(ds)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2$. Find the metric on the chart $C_N$. What is the relation to the ordinary Euclidean metric on the plane?