Physics 464/511  Homework #3  
Due: Oct. 3, 2016 at 5:00 P. M.

1. Consider the unit sphere \( S^2 \). Embedded in three dimensional Euclidean space \( \mathbb{R}^3 \) with cartesian coordinates, this is the set \( \{(x, y, z)|x^2 + y^2 + z^2 = 1\} \), but as it is a two dimensional space, we usually describe it with spherical polar coordinates \( \theta, \phi \), fixing the radius \( r = 1 \), with \( z = \cos \theta \) and \( (x, y) = \sin \theta (\cos \phi, \sin \phi) \). The coordinates \( \theta \) and \( \phi \) are well-defined, however, only for \( \theta \neq 0 \) and \( \neq \pi \). So we have a chart \( C \) for \( U \) the unit sphere with the north and south pole removed.

The unit sphere has a natural metric induced from the Euclidean metric of \( \mathbb{R}^3 \), with which we are very familiar:

\[
(ds)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2,
\]

or \( g_{\theta \theta} = 1, g_{\phi \phi} = \sin^2 \theta, g_{\theta \phi} = 0 \).

(a) Calculate the Christoffel symbol \( \Gamma^j_{\kappa \ell} \)

(b) The shortest path (on the sphere) between two points is determined by the geodesic equations. Find the equations which determine \( \theta(s) \) and \( \phi(s) \), where the \( s \) parameter is the path length. Show that one of these second order equations is essentially the conservation of \( L_z \).

(c) It is easier to guess the solution from the knowledge that the answer is a great circle, the intersection of the sphere with a plane though the origin, rather than solving the equations from scratch. Show that this is correct, satisfying the geodesic equations.

2. Consider the unit sphere \( S^2 \) as the set of point \( \{(x, y, z)|x^2 + y^2 + z^2 = 1\} \) in \( \mathbb{R}^3 \). This is really a two dimensional manifold, so it needs to be considered as such. Consider the two open sets \( U_N \) which is the sphere minus the South pole, and \( U_S \), the sphere minus the North pole. Define the charts \( C_N \) and \( C_S \).
by considering the equatorial plane $\mathcal{E}$. For each point $P \in \mathcal{U}_N$ on the sphere, draw straight lines (in $\mathbb{R}^3$) through the South pole and $P$, and define $\phi_N(P)$ to be the point $(x_N, y_N)$ at which the line intersects the equatorial plane. Similarly define $\phi_S(P)$ to be the point in equatorial plane on the straight line through the North pole and $P$, but call that point $(x_S, -y_S)$. [The reason for inverting $y$ is to keep $dx_S \wedge dy_S$ pointing to the outside of the sphere.]

(a) Find the coordinates of $\phi_N(P)$ in terms of the $(x, y, z)$ of $P$ in $\mathbb{R}^3$, and the inverse relation.

(b) Do the same for $\phi_S(P)$, and find the transition function

$$\phi_{SN} = \phi_S \circ \phi_N^{-1} = \phi_s \left( \phi_N^{-1}(x_N, y_N) \right).$$

(c) The metric on the sphere is $(ds)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2$. Find the metric on the chart $C_N$. What is the relation to the ordinary Euclidean metric on the plane?